

Eigenvector statistics of the product of Ginibre matrices

Zdzisław Burda^{*} and Bartłomiej J. Spisak[†]

AGH University of Science and Technology, Faculty of Physics and Applied Computer Science, al. Mickiewicza 30, 30-059 Kraków, Poland

Pierpaolo Vivo[‡]

Department of Mathematics, King's College London, Strand WC2R 2LS, London, U.K.

We develop a method to calculate left-right eigenvector correlations of the product of m independent $N \times N$ complex Ginibre matrices. For illustration, we present explicit analytical results for the vector overlap for a couple of examples for small m and N . We conjecture that the integrated overlap between left and right eigenvectors is given by the formula $O = 1 + (m/2)(N - 1)$ and support this conjecture by analytical and numerical calculations. We derive an analytical expression for the limiting correlation density as $N \rightarrow \infty$ for the product of Ginibre matrices as well as for the product of elliptic matrices. In the latter case, we find that the correlation function is independent of the eccentricities of the elliptic laws.

Keywords: random matrix theory, non-hermitian, planar diagram enumeration

^{*} zdzislaw.burda@agh.edu.pl

[†] bjs@agh.edu.pl

[‡] pierpaolo.vivo@kcl.ac.uk

I. INTRODUCTION

Products of random matrices have continuously attracted attention since the sixties [1–5]. They are of relevance in many fields of mathematics, physics and engineering including dynamical systems [2, 6], disordered systems [7–9], statistical mechanics [10], quantum mechanics [11], quantum transport and mesoscopic systems [12, 13], hidden Markov models [14], image processing [15], quantum chromodynamics [16], wireless telecommunication [17, 18], quantitative finance [19–21] and many others [22]. Recently, an enormous progress has been made in the understanding of macroscopic [23–44] and microscopic [45–66] statistics of eigenvalues and singular values as well as of Lyapunov spectra for products of random matrices [67–77]. In contrast, not much has been learned about the eigenvector statistics of the products of random matrices so far. In this paper, we address this problem by considering a correlation function for eigenvectors of the product of Ginibre matrices. More precisely, we study the overlap between left and right eigenvectors for finite N and for $N \rightarrow \infty$. In the first part of the paper, we adapt ideas developed in [78, 79] to the product of random matrices by using the generalized Schur decomposition [45] for finite N , while in the second part we combine the generalized Green's function method [81–84] with linearization (subordination) [10, 28, 37] to derive the limiting law for the overlap for $N \rightarrow \infty$.

II. DEFINITIONS

Consider a diagonalizable matrix X over the field of complex numbers. Let $\{\Lambda_\alpha\}$ be the eigenvalues of X . The corresponding left eigenvectors $\langle L_\alpha|$ and right eigenvectors $|R_\alpha\rangle$ satisfy the relations

$$X |R_\alpha\rangle = \Lambda_\alpha |R_\alpha\rangle, \quad \langle L_\alpha| X = \langle L_\alpha| \Lambda_\alpha. \quad (1)$$

Note that the Hermitian conjugate of the second equation has the form: $X^\dagger |L_\alpha\rangle = \bar{\Lambda}_\alpha |L_\alpha\rangle$, where the symbol 'bar' denotes the complex conjugation of Λ_α . The eigenvectors fulfill the bi-orthogonality and closure relations in the form

$$\langle L_\alpha | R_\beta \rangle = \delta_{\alpha\beta}, \quad \sum_\alpha |L_\alpha\rangle \langle R_\alpha| = 1. \quad (2)$$

The two relations are invariant with respect to the scale transformation

$$|R_\alpha\rangle \rightarrow c_\alpha |R_\alpha\rangle, \quad \langle L_\alpha| \rightarrow \langle L_\alpha| c_\alpha^{-1} \quad (3)$$

with arbitrary non-zero coefficients c_α 's. According to Refs. [78, 79], an overlap of the left and right eigenvectors is defined in the following way

$$O_{\alpha\beta} = \langle L_\alpha | L_\beta \rangle \langle R_\beta | R_\alpha \rangle. \quad (4)$$

By construction, the quantity $O_{\alpha\beta}$ is invariant with respect to the scale transformation given by Eq. (3) and consequently does not depend on the vector normalizations.

If X is a random matrix, one defines averages over the ensemble

$$\langle O_{\alpha\beta} \rangle = \int d\mu(X) O_{\alpha\beta}, \quad (5)$$

where $d\mu(X)$ is the probability measure for the random matrix in question. The dependence of $O_{\alpha\beta}$ on X is suppressed in the notation. We used this notation throughout the paper also for other observables which depend on random matrices. The global diagonal overlap averaged over the ensemble is given by

$$O = \left\langle \frac{1}{N} \sum_{\alpha=1}^N O_{\alpha\alpha} \right\rangle, \quad (6)$$

while the global off-diagonal one is expressed by the formula

$$O_{off} = \left\langle \frac{2}{N(N-1)} \sum_{\alpha < \beta} O_{\alpha\beta} \right\rangle. \quad (7)$$

We are interested here in invariant random matrices for which the probability measure is invariant with respect to the similarity transformation $X \rightarrow UXU^{-1}$, where U is a unitary matrix. In particular, this invariance implies that $\langle O_{\alpha\alpha} \rangle = \langle O_{11} \rangle$ and $\langle O_{\alpha\beta} \rangle = \langle O_{12} \rangle$ for any α and β . It follows that

$$O = \langle O_{11} \rangle, \quad O_{off} = \langle O_{12} \rangle. \quad (8)$$

We can also define the local diagonal overlap density by the formula

$$O(z) = \left\langle \frac{1}{N} \sum_{\alpha=1}^N O_{\alpha\alpha} \delta(z - \Lambda_\alpha) \right\rangle = \langle O_{11} \delta(z - \Lambda_1) \rangle, \quad (9)$$

and the off-diagonal one by

$$\begin{aligned} O_{off}(z, w) &= \left\langle \frac{2}{N(N-1)} \sum_{\alpha < \beta} O_{\alpha\beta} \delta(z - \Lambda_\alpha) \delta(w - \Lambda_\beta) \right\rangle \\ &= \langle O_{12} \delta(z - \Lambda_1) \delta(w - \Lambda_2) \rangle. \end{aligned} \quad (10)$$

Clearly, the diagonal global overlap is equal to the integrated overlap density given by Eq. (9), i.e.

$$O = \int d^2 z O(z). \quad (11)$$

III. PRODUCT OF GINIBRE MATRICES

Consider the product

$$X = X_1 X_2 \cdots X_m \quad (12)$$

of m independent identically distributed $N \times N$ Ginibre random matrices [85] with complex entries. The probability measure factorizes and can be written as a product of measures for individual Ginibre matrices

$$d\mu(X) \equiv d\mu(X_1, X_2, \dots, X_m) = d\mu(X_1) d\mu(X_2) \cdots d\mu(X_m), \quad (13)$$

each of which is given by

$$d\mu(X_i) = (\pi\sigma^2)^{-N^2} e^{-\frac{1}{\sigma^2} \text{Tr} X_i X_i^\dagger} DX_i, \quad (14)$$

where σ is a scale parameter, and $DX_i = \prod_{\alpha\beta} d\text{Re} X_{i,\alpha\beta} d\text{Im} X_{i,\alpha\beta}$. According to Eq. (9) the local diagonal overlap density can be calculated with respect to the measure $d\mu(X)$ in the following way

$$O(z) = \int d\mu(X) O_{11} \delta(z - \Lambda_1), \quad (15)$$

where Λ_α 's correspond to the eigenvalues of the product X (12). An analogous formula holds for the off-diagonal density. In the calculations we set $\sigma = 1$. One can easily transform the result to other values of σ by using the formula

$$O_\sigma(z) = \frac{1}{\sigma^{2m}} O_{\sigma=1} \left(\frac{z}{\sigma^m} \right), \quad (16)$$

which merely corresponds to the scale transformation of all Ginibre matrices $X_i \rightarrow \sigma X_i$ in the product (12). Later, when discussing the limiting laws for $N \rightarrow \infty$ we will choose $\sigma = N^{-1/2}$. This choice of the scale parameter σ will ensure the existence of the limiting eigenvalue density on a compact support being the unit disk in the complex plane.

IV. CALCULATIONS OF THE OVERLAP FOR FINITE N

In order to calculate the global left-right vector overlap, defined by Eq. (4), for the product of Ginibre matrices (12), we will change the parametrization of the matrices X_i 's using the generalized Schur decomposition [45]

$$X_i = U_{i-1} \tau_i U_i^\dagger, \quad (17)$$

for $i = 1, \dots, m$, where U_i are unitary matrices from the unitary group $U(N)$, and τ_i are upper triangular matrices of size $N \times N$. We use a cyclic indexing $U_i \equiv U_{m+i}$, in particular $U_0 \equiv U_m$. Sometimes it is convenient to express each τ_i as a sum

of a diagonal matrix λ_i and a strictly upper triangular one t_i , namely

$$\tau_i = \lambda_i + t_i = \begin{pmatrix} \lambda_{i,1} & t_{i,12} & t_{i,13} & \cdots & t_{i,1N} \\ 0 & \lambda_{i,2} & t_{i,23} & \cdots & t_{i,2N} \\ & & & \ddots & \\ 0 & 0 & 0 & \lambda_{i,N-1} & t_{i,N-1N} \\ 0 & 0 & 0 & \cdots & \lambda_{i,N} \end{pmatrix}. \quad (18)$$

In this representation, the product X is unitarily equivalent to a matrix \mathcal{T} that is $X = U_m \mathcal{T} U_m^\dagger$, where

$$\mathcal{T} = \tau_1 \tau_2 \cdots \tau_m. \quad (19)$$

The matrix \mathcal{T} has also an upper triangular form

$$\mathcal{T} = \Lambda + T = \begin{pmatrix} \Lambda_1 & T_{12} & T_{13} & \cdots & T_{1N} \\ 0 & \Lambda_2 & T_{23} & \cdots & T_{2N} \\ & & & \ddots & \\ 0 & 0 & 0 & \Lambda_{N-1} & T_{N-1N} \\ 0 & 0 & 0 & \cdots & \Lambda_N \end{pmatrix}. \quad (20)$$

The diagonal elements of \mathcal{T} are given by

$$\mathcal{T}_\alpha \equiv \Lambda_\alpha = \lambda_{1,\alpha} \lambda_{2,\alpha} \cdots \lambda_{m,\alpha}, \quad (21)$$

and the off-diagonal ones by

$$\mathcal{T}_{\alpha\nu} = \sum_{\alpha \leq \beta \leq \dots \leq \nu} \tau_{1,\alpha\beta} \tau_{2,\beta\gamma} \cdots \tau_{m,\mu\nu}. \quad (22)$$

Any instance of $\tau_{i,\alpha\alpha}$ with two identical Greek indices can be replaced by $\lambda_{i,\alpha}$ and of $\tau_{i,\alpha\beta}$ with two different Greek indices by $t_{i,\alpha\beta}$ in the last formula. One can also express the integration measure in terms of U 's, λ 's and t 's. Since one is interested in invariant observables, the U 's can be integrated out. For the scale parameter $\sigma = 1$ one gets [45]

$$d\mu(\lambda, t) = Z^{-1} |\Delta(\Lambda)|^2 \prod_{i,\alpha} e^{-|\lambda_{i,\alpha}|^2} d^2 \lambda_{i,\alpha} \prod_{j,\beta < \gamma} \frac{1}{\pi} e^{-|t_{j,\beta\gamma}|^2} d^2 t_{j,\beta\gamma}, \quad (23)$$

where the normalization factor Z is given by the formula

$$Z = N! [\pi^N 1! 2! \cdots (N-1)!]^m, \quad (24)$$

and the Vandermonde determinant $\Delta(\Lambda)$ for the product $X = X_1 X_2 \cdots X_m$ has the form

$$\Delta(\Lambda) = \prod_{\alpha < \beta} (\lambda_{1,\alpha} \lambda_{2,\alpha} \cdots \lambda_{m,\alpha} - \lambda_{1,\beta} \lambda_{2,\beta} \cdots \lambda_{m,\beta}) = \prod_{\alpha < \beta} (\Lambda_\alpha - \Lambda_\beta). \quad (25)$$

The square of the determinant in Eq. (23) comes from the Jacobian of the transformation (17).

The next step is to express the observables in terms of t 's and λ 's. For example, to calculate the diagonal overlap density [cf. Eq. (15)], we have to find $O_{11} = O_{11}(t, \lambda)$ and to integrate over t 's and λ 's with the Dirac's delta constraint

$$O(z) = \int d\mu(\lambda, t) O_{11}(t, \lambda) \delta(z - \Lambda_1), \quad (26)$$

while for the global overlap $O = \int d\mu(\lambda, t) O_{11}(t, \lambda)$. The measure $d\mu(\lambda, t)$ factorizes $d\mu(\lambda, t) = d\mu(\lambda) d\mu(t)$. One can first integrate over t 's. This is a Gaussian integral and can be easily performed. After this integration, only the dependence on λ 's is left

$$O_{11}(\lambda) = \int d\mu(t) O_{11}(t, \lambda). \quad (27)$$

The last step is to integrate over λ 's with the measure given by Eq. (23)

$$O(z) = Z^{-1} \int d\mu(\lambda) |\Delta(\Lambda)|^2 e^{-\sum_{i,\alpha} |\lambda_{i,\alpha}|^2} O_{11}(\lambda) \delta(z - \Lambda_1), \quad (28)$$

where as before Λ_α 's stand for $\Lambda_\alpha = \lambda_{1,\alpha} \lambda_{2,\alpha} \cdots \lambda_{m,\alpha}$. We will do this below. First we have to find the function $O_{11}(t, \lambda)$. This can be done as follows. We choose the basis in which the product matrix X is equal to \mathcal{T} . Such a basis exists since the two matrices are unitarily equivalent. In this basis, the right vector $|R_\alpha\rangle$ is represented as a column vector with one in the position α and zeros in the other positions: $|R_1\rangle = (1, 0, 0, \dots)^T$, $|R_2\rangle = (0, 1, 0, \dots)^T$. The vectors are written here as transposes of row vectors to save space. Denote the elements of the first left eigenvector $\langle L_1| = (B_1, B_2, \dots)$. Using the bi-orthogonality relation (2), one finds the following recursion relation for B_β 's [78, 79]

$$B_\beta = \frac{1}{\Lambda_1 - \Lambda_\beta} \sum_{\alpha=1}^{\beta-1} B_\alpha T_{\alpha\beta}. \quad (29)$$

The recursion is initiated by $B_1 = 1$. It gives

$$\begin{aligned} B_1 &= 1, \\ B_2 &= \frac{T_{12}}{\Lambda_1 - \Lambda_2}, \\ B_3 &= \frac{T_{13}}{\Lambda_1 - \Lambda_3} + \frac{T_{12}T_{23}}{(\Lambda_1 - \Lambda_2)(\Lambda_1 - \Lambda_3)}, \\ B_4 &= \frac{T_{14}}{\Lambda_1 - \Lambda_4} + \frac{T_{12}T_{24}}{(\Lambda_1 - \Lambda_2)(\Lambda_1 - \Lambda_4)} + \frac{T_{13}T_{34}}{(\Lambda_1 - \Lambda_3)(\Lambda_1 - \Lambda_4)} + \\ &\quad + \frac{T_{12}T_{23}T_{34}}{(\Lambda_1 - \Lambda_2)(\Lambda_1 - \Lambda_3)(\Lambda_1 - \Lambda_4)}, \quad \text{etc.} \end{aligned} \quad (30)$$

The element O_{11} of the overlap matrix is related to B 's as

$$O_{11} = \sum_{\alpha=1}^N |B_\alpha|^2 \quad (31)$$

and B 's depend on t 's and λ 's through T 's and Λ 's. Combining Eqs. (30),(31) with Eq. (26) we obtain an explicit form of the integral over t 's and λ 's which can be done. We will give a couple of examples below.

V. EXAMPLES

Let us first illustrate the calculations for $N = 2$, $m = 2$ and $\sigma = 1$ - that is for the product of two 2×2 Ginibre matrices. Firstly, we express T_{12} in terms of t 's and λ 's as follows

$$\mathcal{T} = \begin{pmatrix} \lambda_{1,1} & t_{1,12} \\ 0 & \lambda_{1,2} \end{pmatrix} \begin{pmatrix} \lambda_{2,1} & t_{2,12} \\ 0 & \lambda_{2,2} \end{pmatrix} = \begin{pmatrix} \Lambda_1 & T_{12} \\ 0 & \Lambda_2 \end{pmatrix}. \quad (32)$$

This gives $T_{12} = \lambda_{1,1}t_{2,12} + t_{1,12}\lambda_{2,2}$ and $\Lambda_\alpha = \lambda_{1,\alpha}\lambda_{2,\alpha}$ for $\alpha = 1, 2$. Thus we have

$$O_{11}(t, \lambda) = 1 + \frac{|T_{12}|^2}{|\Lambda_1 - \Lambda_2|^2} = 1 + \frac{|\lambda_{1,1}t_{2,12} + t_{1,12}\lambda_{2,2}|^2}{|\lambda_{1,1}\lambda_{2,1} - \lambda_{1,2}\lambda_{2,2}|^2}. \quad (33)$$

According to Eq. (27), the integration over t 's leads to the following result

$$O_{11}(\lambda) = 1 + \frac{|\lambda_{1,1}|^2 + |\lambda_{2,2}|^2}{|\lambda_{1,1}\lambda_{2,1} - \lambda_{1,2}\lambda_{2,2}|^2}. \quad (34)$$

Now we have to compute the integral over λ 's given by Eq. (28), namely

$$O(z) = \frac{1}{2\pi^4} \int (|\lambda_{1,1}\lambda_{2,1} - \lambda_{1,2}\lambda_{2,2}|^2 + |\lambda_{1,1}|^2 + |\lambda_{2,2}|^2) \delta(z - \lambda_{1,1}\lambda_{2,1}) \prod_{i,\alpha} e^{-|\lambda_{i,\alpha}|^2} d^2\lambda_{i,\alpha}. \quad (35)$$

We first integrate over the λ 's that do not appear in the Dirac's delta, that is $\lambda_{1,2}$ and $\lambda_{2,2}$. These integrals are in general of the Gaussian type combined with a power function, i.e. $\int d^2z |z|^{2k} \exp(-|z|^2) = \pi k!$. As a result of the integration, we obtain

$$O(z) = \frac{1}{2\pi^2} \int (|z|^2 + 2 + |\lambda_{1,1}|^2) \delta(z - \lambda_{1,1}\lambda_{2,1}) e^{-|\lambda_{1,1}|^2 - |\lambda_{2,1}|^2} d^2\lambda_{1,1} d^2\lambda_{2,1}. \quad (36)$$

Now we integrate over $\lambda_{2,1}$. We use the scaling property of the Dirac's delta $\delta(a(z - z_0)) = (1/|a|^2)\delta(z - z_0)$ to get

$$O(z) = \frac{1}{2\pi^2} \int \frac{|z|^2 + 2 + |\lambda_{1,1}|^2}{|\lambda_{1,1}|^2} \exp\left(-|\lambda_{1,1}|^2 - \frac{|z|^2}{|\lambda_{1,1}|^2}\right) d^2\lambda_{1,1} . \quad (37)$$

The integral over $\lambda_{1,1}$ can be conveniently done in polar coordinates, $\lambda_{1,1} = \sqrt{x} \exp(i\phi)$

$$O(z) = \frac{1}{2\pi} \int_0^\infty \frac{|z|^2 + 2 + x}{x} \exp\left(-x - \frac{|z|^2}{x}\right) dx , \quad (38)$$

yielding

$$O(z) = \frac{1}{\pi} \left[(2 + |z|^2) K_0(2|z|) + |z| K_1(2|z|) \right] , \quad (39)$$

where K_ν denotes the modified Bessel function of the second kind. The global overlap is

$$O = \int d^2z O(z) = 2 . \quad (40)$$

The overlap density depends on the modulus $|z|$. It is convenient to represent this quantity as a radial function in the variable $r = |z|$,

$$O_{rad}(r) = 2\pi r O(r) . \quad (41)$$

Clearly $O_{rad}(r)dr$ is equal to the overlap density integrated over the annulus $r \leq |z| \leq r + dr$. In our case we have

$$O_{rad}(r) = 2r(2 + r^2)K_0(2r) + 2r^2 K_1(2r) . \quad (42)$$

In principle, one may repeat the calculation for any N and m . All integrals except those over the λ 's appearing in the argument of the Dirac's delta, i.e. $\delta(z - \lambda_{1,1} \cdots \lambda_{1,m})$ are Gaussian and can be done explicitly. The integrals over λ 's from the Dirac's delta generate instead Meijer G-functions due to the multiplicative constraint [80]. Let us illustrate it for the product of three 2×2 Ginibre matrices. The calculation goes as before. The element T_{12} of the \mathcal{T} matrix is

$$T_{12} = \lambda_{1,1}\lambda_{2,1}t_{3,12} + \lambda_{1,1}t_{2,12}\lambda_{3,2} + t_{1,12}\lambda_{2,2}\lambda_{3,2} , \quad (43)$$

and the diagonal elements are $\Lambda_1 = \lambda_{1,1}\lambda_{2,1}\lambda_{3,1}$, $\Lambda_2 = \lambda_{1,2}\lambda_{2,2}\lambda_{3,2}$. Hence, the counterpart of Eq. (33) is

$$O_{11}(\lambda, t) = 1 + \frac{|\lambda_{1,1}\lambda_{2,1}t_{3,12} + \lambda_{1,1}t_{2,12}\lambda_{3,2} + t_{1,12}\lambda_{2,2}\lambda_{3,2}|^2}{|\lambda_{1,1}\lambda_{2,1}\lambda_{3,1} - \lambda_{1,2}\lambda_{2,2}\lambda_{3,2}|^2} . \quad (44)$$

Integrating over t 's we get

$$O_{11}(\lambda) = 1 + \frac{|\lambda_{1,1}\lambda_{2,1}|^2 + |\lambda_{1,1}\lambda_{3,2}|^2 + |\lambda_{2,2}\lambda_{3,2}|^2}{|\lambda_{1,1}\lambda_{2,1}\lambda_{3,1} - \lambda_{1,2}\lambda_{2,2}\lambda_{3,2}|^2} , \quad (45)$$

and over the λ 's (except those in the Dirac's delta)

$$O(z) = \frac{1}{2\pi^6} \int (|z|^2 + 2 + |\lambda_{1,1}\lambda_{2,1}|^2 + |\lambda_{1,1}|^2) \delta(z - \lambda_{1,1}\lambda_{2,1}\lambda_{3,1}) e^{-|\lambda_{1,1}|^2 - |\lambda_{2,1}|^2 - |\lambda_{3,1}|^2} d^2\lambda_{1,1} d^2\lambda_{2,1} d^2\lambda_{3,1} . \quad (46)$$

Next, we integrate over $\lambda_{3,1}$ and use polar coordinates for $\lambda_{1,1} = \sqrt{x_1} \exp(i\phi_1)$ and $\lambda_{2,1} = \sqrt{x_2} \exp(i\phi_2)$. We eventually obtain

$$O(z) = \frac{1}{2\pi} \int_0^\infty \int_0^\infty \frac{|z|^2 + 2 + x_1x_2 + x_1}{x_1x_2} \exp\left(-x_1 - x_2 - \frac{|z|^2}{x_1x_2}\right) dx_1 dx_2 , \quad (47)$$

which yields the radial function

$$O_{rad}(r) = r^2 G_{03}^{30} \left(\begin{matrix} - \\ -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{matrix} \middle| r^2 \right) + 2r G_{03}^{30} \left(\begin{matrix} - \\ 0, 0, 0 \end{matrix} \middle| r^2 \right) + r G_{03}^{30} \left(\begin{matrix} - \\ 0, 0, 1 \end{matrix} \middle| r^2 \right) + r G_{03}^{30} \left(\begin{matrix} - \\ 1, 1, 1 \end{matrix} \middle| r^2 \right) . \quad (48)$$

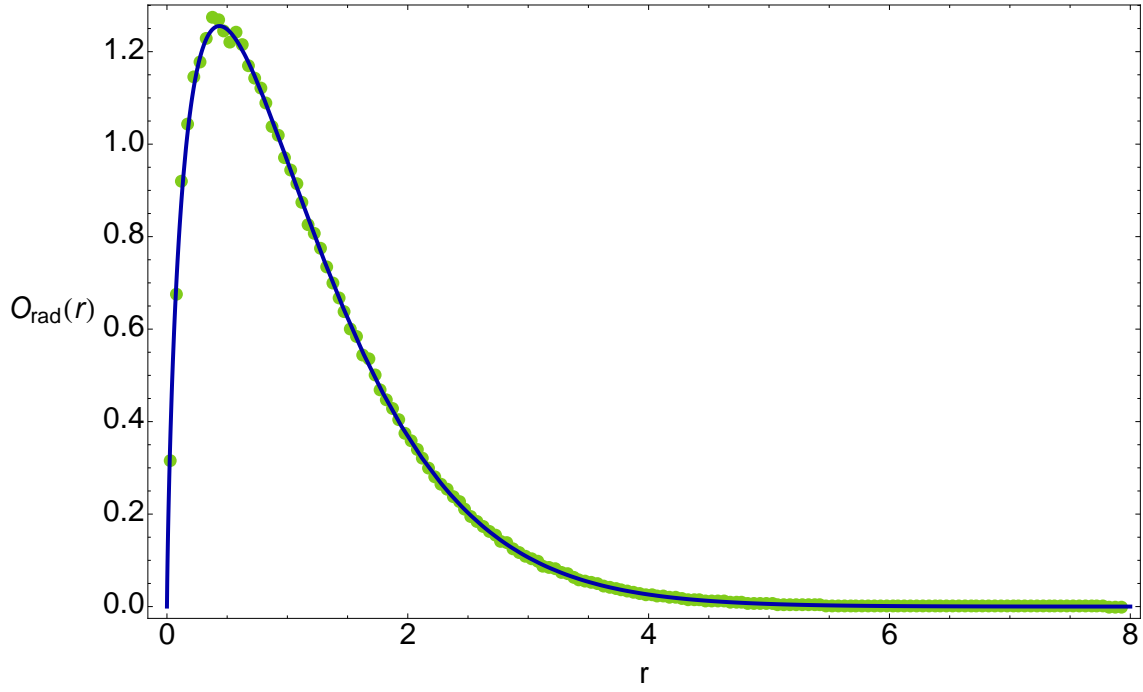


FIG. 1. Overlap density for $N = 2$ and $m = 2$: theoretical prediction given by Eq. (42) (solid line) and numerical histogram (points) generated in Monte Carlo simulations of 10^6 products of two 2×2 Ginibre matrices.

One finds that the global overlap for $N = 2$ and $m = 3$ is

$$O = \int d^2 z O(z) = \int_0^\infty O_{rad}(r) dr = \frac{5}{2}. \quad (49)$$

One may repeat the calculations for larger N and larger m . The integrals one has to do are elementary but the bookkeeping gets complex and the calculations become tedious. For example, for $N = 3$ and $m = 2$ one has to sum three terms depending on the coefficients B_1 , B_2 and B_3 as it stems from Eq. (30), which depend on λ 's and t 's through Λ 's and T 's: $T_{12} = \lambda_{1,1}t_{2,12} + t_{1,12}\lambda_{2,2}$, $T_{13} = \lambda_{1,1}t_{2,13} + t_{1,12}t_{2,23} + t_{1,13}\lambda_{2,3}$ and $T_{23} = \lambda_{1,2}t_{2,23} + t_{1,23}\lambda_{2,3}$. Integrals over t 's can be done in an algebraic way using Wick's theorem and the following two-point functions

$$\langle t_{i,\alpha\beta} \bar{t}_{j,\mu\nu} \rangle_t = \delta_{ij} \delta_{\alpha\mu} \delta_{\beta\nu}, \quad \langle t_{i,\alpha\beta} t_{j,\mu\nu} \rangle_t = 0. \quad (50)$$

where the symbol $\langle t_{i,\alpha\beta} \bar{t}_{j,\mu\nu} \rangle_t$ is to be understood as follows

$$\langle t_{i,\alpha\beta} \bar{t}_{j,\mu\nu} \rangle_t = \int t_{i,\alpha\beta} \bar{t}_{j,\mu\nu} \prod_{k,\eta < \gamma} \frac{1}{\pi} e^{-|t_{k,\eta\gamma}|^2} d^2 t_{k,\eta\gamma}. \quad (51)$$

We skip the calculations and give the final result which reads

$$O_{rad}(r) = \frac{1}{3} r (r^4 + 8r^2 + 12) K_0(2r) + \frac{1}{3} (2r^4 + 8r^2) K_1(2r) \quad (52)$$

and

$$O = \int O_{rad}(r) dr = 3. \quad (53)$$

In Figs. 1, 2 and 3, we show the theoretical predictions for the radial profile of the overlap densities and the corresponding histograms from Monte Carlo simulations for $N = 2, m = 2$ [cf. Eq. (42)], $N = 2, m = 3$ [cf. Eq. (48)] and $N = 3, m = 2$ [cf. Eq. (52)], respectively. We see that the Monte Carlo data follow the theoretical curves.

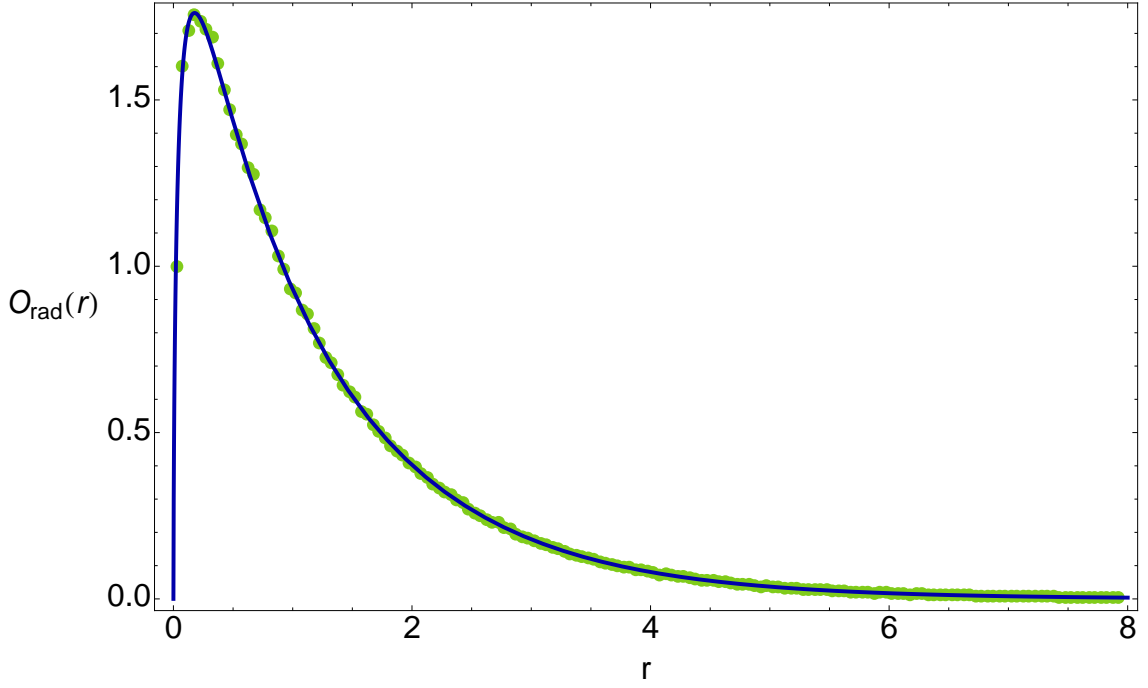


FIG. 2. Overlap density for $N = 2$ and $m = 3$: theoretical prediction given by Eq. (48) (solid line) and numerical histogram (points) generated in Monte Carlo simulations of 10^6 products of three 2×2 Ginibre matrices.

VI. CONJECTURE

The calculations of the global density are slightly easier because there is no Dirac's delta $\delta(z - \Lambda_1)$ in the integrand. They are particularly simple for $N = 2$. In this case

$$T_{12} = \sum_{k=1}^m t_{k,12} \prod_{j=1}^{k-1} \lambda_{j,1} \prod_{j=k+1}^N \lambda_{j,2}, \quad (54)$$

and after inserting this into Eq. (33) and integrating the t 's, one obtains

$$O = 1 + \frac{1}{2\pi^{2m}} \sum_{k=1}^m \int \prod_{j=1}^{k-1} |\lambda_{j,1}|^2 \prod_{j=k+1}^N |\lambda_{j,2}|^2 \prod_{i=1}^N e^{-|\lambda_{i,1}|^2 - |\lambda_{i,2}|^2} d^2 \lambda_{i,1} d^2 \lambda_{i,2}. \quad (55)$$

Each integral over λ is either of the form $\int |z|^2 \exp(-|z|^2) d^2 z = \pi$ or $\int \exp(-|z|^2) d^2 z = \pi$, so all together the integration over λ 's gives the factor π^{2m} which cancels the pre-factor π^{-2m} yielding

$$O = 1 + \frac{m}{2}. \quad (56)$$

Now, consider the case $m = 1$ for any N . One finds [79]

$$O = \frac{1}{Z} \int \prod_{\alpha=1}^{N-1} \left(1 + \frac{1}{|\lambda_N - \lambda_\alpha|^2} \right) |\Delta(\lambda)|^2 \prod_{\alpha=1}^N e^{-|\lambda_\alpha|^2} d^2 \lambda_\alpha, \quad (57)$$

with $Z = \pi^N 1!2! \cdots N!$ as it stems from Eq. (24). The integral yields

$$O = 1 + \frac{1}{2}(N-1). \quad (58)$$

See Appendix A for the proof.

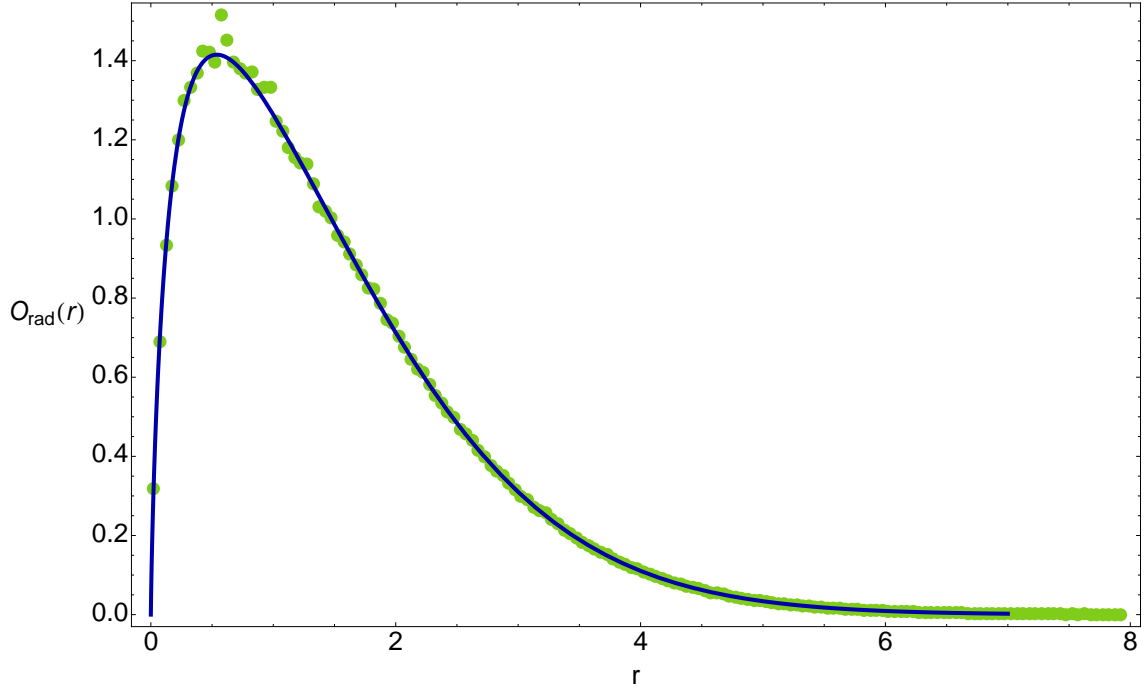


FIG. 3. Overlap density for $N = 3$ and $m = 2$: theoretical prediction given by Eq. (52) (solid line) and numerical histogram (points) generated in Monte Carlo simulations of 10^6 products of two 3×3 Ginibre matrices.

The results given by Eqs. (56) and (58) suggest that O grows linearly with m and N , hence it is tempting to conjecture that for any m and N the global overlap is given by the formula

$$O = 1 + \frac{m}{2}(N - 1). \quad (59)$$

The result given by Eq. (53) is in agreement with this formula and Monte Carlo simulations fully corroborate this conjecture as shown in Fig. 4.

VII. LARGE N LIMIT

We now consider the limit $N \rightarrow \infty$. We set the width parameter $\sigma^2 = 1/N$ in the measure (14). The limit $N \rightarrow \infty$ has to be taken carefully since we expect $O(z)$ to grow with N as it results from Eq. (59). It is convenient to define the growth rate of the overlap density as

$$o(z) = \frac{O(z)}{N}, \quad (60)$$

which is expected to approach an N -independent function $o(z)$ for $N \rightarrow \infty$, such that $\int d^2z o(z) = m/2$ as follows from Eq. (59).

In the calculations, we shall use the method [28] that was previously used to calculate the limiting eigenvalue density

$$\rho(z) = \lim_{N \rightarrow \infty} \left\langle \frac{1}{N} \sum_{j=1}^N \delta(z - \Lambda_j) \right\rangle. \quad (61)$$

The method is based on the generalized Green's function [81–83]

$$\widehat{G}(z, \epsilon) = \left\langle \left(\begin{pmatrix} z \mathbb{1}_N - X & \epsilon \mathbb{1}_N \\ -\bar{\epsilon} \mathbb{1}_N & \bar{z} \mathbb{1}_N - X^\dagger \end{pmatrix}^{-1} \right) \right\rangle, \quad (62)$$

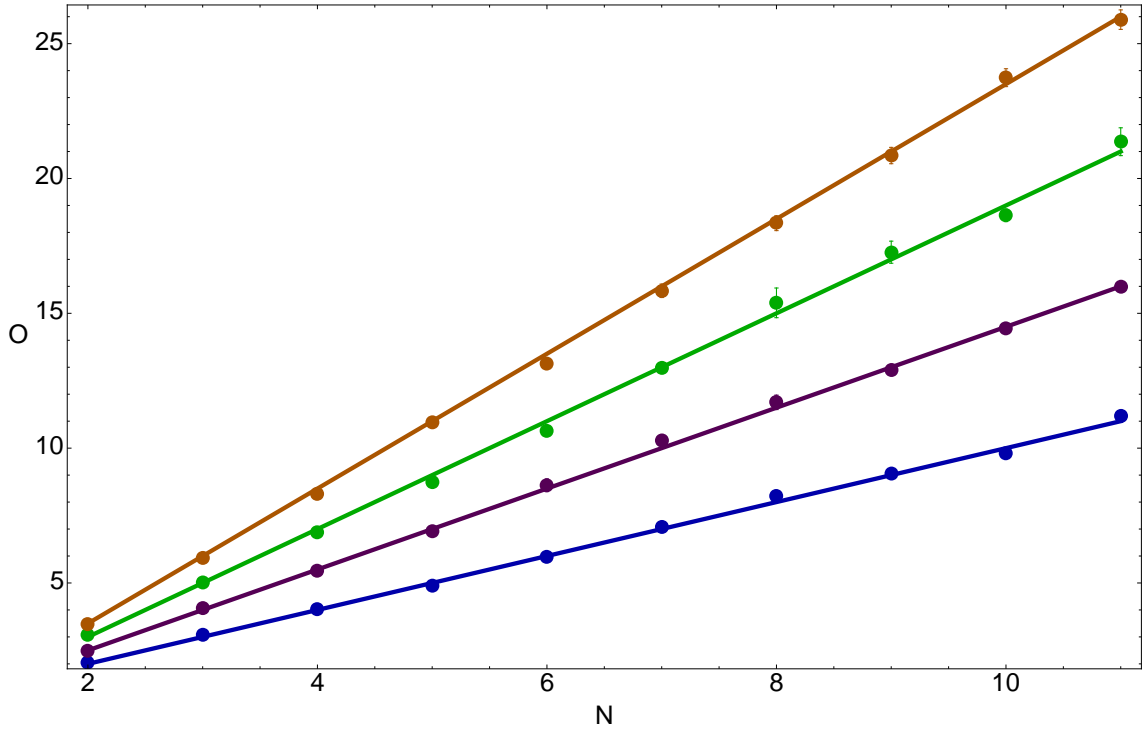


FIG. 4. Conjectured form of the overlap (59) (solid lines) for $m = 2, 3, 4, 5$ and $N = 2, \dots, 11$ and numerical histograms (points) generated in Monte Carlo simulations, each for 10^4 instances.

which consists of $N \times N$ blocks $G_{\alpha\beta}$

$$\hat{G}(z, \epsilon) = \begin{pmatrix} G_{11}(z, \epsilon) & G_{12}(z, \epsilon) \\ G_{21}(z, \epsilon) & G_{22}(z, \epsilon) \end{pmatrix}. \quad (63)$$

For clarity, the symbol 'hat' is reserved for matrices with a superimposed block structure. By defining the block-trace Tr_b as a matrix of traces of individual blocks

$$\text{Tr}_b \hat{G} = \begin{pmatrix} \text{Tr } G_{11} & \text{Tr } G_{12} \\ \text{Tr } G_{21} & \text{Tr } G_{22} \end{pmatrix}, \quad (64)$$

one can project the $2N \times 2N$ matrix \hat{G} onto a 2×2 matrix \hat{g}

$$\hat{g}(z) = \begin{pmatrix} g_{11}(z) & g_{12}(z) \\ g_{21}(z) & g_{22}(z) \end{pmatrix} = \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \text{Tr}_b \hat{G}(z, \epsilon). \quad (65)$$

The elements of this matrix are related to each other, $g_{22}(z) = \bar{g}_{11}(z)$ and $g_{21}(z) = -\bar{g}_{12}(z)$ [44], so we have

$$\hat{g}(z) = \begin{pmatrix} g(z) & \gamma(z) \\ -\bar{\gamma}(z) & \bar{g}(z) \end{pmatrix}. \quad (66)$$

The eigenvalue density is related to the diagonal element [81–83]

$$\rho(z) = \frac{1}{\pi} \frac{\partial g(z)}{\partial \bar{z}}, \quad (67)$$

and the growth rate of the overlap to the off-diagonal one [84]

$$o(z) = \frac{1}{\pi} |\gamma(z)|^2. \quad (68)$$

For large N , the leading contribution to the overlap grows linearly with N : $O(z) \sim No(z)$.

Equations (67) and (68) are general and can be applied to any random matrix provided the Green's function $\hat{g}(z)$ can be calculated. So the goal is now to calculate the Green's function for the problem in question. To this end, we use the planar diagrams enumeration technique [86–88].

VIII. DYSON-SCHWINGER EQUATIONS

Enumeration of planar Feynman diagrams is a method to derive the large N limit for matrix models [86–88]. The method is based on a field-theoretical representation of multidimensional integrals in terms of Feynman diagrams. One is interested in calculating the Green's function

$$\hat{G}_{AB} = \left\langle \left(\hat{Q} - \hat{X} \right)_{AB}^{-1} \right\rangle, \quad (69)$$

where \hat{Q} is a constant matrix and \hat{X} is the random matrix that is averaged over. Matrix indices are denoted by A and B in the last equation. In this approach, the Green's function plays the role of generating function for connected two-point Feynman diagrams. The contributions from non-planar diagrams are suppressed at least as $1/N$ in the large N limit, so for $N \rightarrow \infty$ only planar diagrams survive in the counting. One can write a set of equations that relate the Green's function \hat{G}_{AB} to a generating function $\hat{\Sigma}_{AB}$ for one-line irreducible diagrams. Such equations are known in the field-theoretical literature as Dyson-Schwinger equations. Here, we are interested only in Gaussian random matrices. In this case, the Dyson-Schwinger equations assume a simple form in the planar limit $N \rightarrow \infty$ [28]

$$\begin{aligned} \hat{G}_{AB} &= \left(\hat{Q} - \hat{\Sigma} \right)_{AB}^{-1}, \\ \hat{\Sigma}_{AD} &= \sum_{BC} \hat{P}_{AB,CD} \hat{G}_{BC}, \end{aligned} \quad (70)$$

where $\hat{P}_{AB,CD}$ represents the propagator

$$\hat{P}_{AB,CD} = \langle \hat{X}_{AB} \hat{X}_{CD} \rangle. \quad (71)$$

The matrix \hat{Q}_{AB} and the propagator $\hat{P}_{AB,CD}$ are inputs to be injected into these equations, while \hat{G}_{AB} and $\hat{\Sigma}_{AB}$ are unknown functions to be determined for the given inputs. In other words, one has first to specify what \hat{Q} and \hat{P} are, and then, using these equations, one can find the Green's function \hat{G} , from there \hat{g} and finally the eigenvalue density $\rho(z)$ [cf. Eq. (67)] and the overlap growth rate $o(z)$ [cf. Eq. (68)].

IX. SINGLE GINIBRE MATRIX

In this section, we review the calculations [82, 84] for a single Ginibre matrix [85]. In the next section, we will then show how to generalize the method to the product of Ginibre matrices [28].

As mentioned before, first one has to identify the matrix \hat{Q} and to calculate the propagator $\hat{P}_{AB,CD}$. The Green's function (62) reads

$$\hat{G}(z, \epsilon) = \left\langle \left(\hat{Q} - \hat{X} \right)^{-1} \right\rangle, \quad (72)$$

with

$$\hat{X} = \begin{pmatrix} X & 0 \\ 0 & X^\dagger \end{pmatrix} \quad (73)$$

and

$$\hat{Q} = \hat{q} \otimes \mathbb{1}_N, \quad (74)$$

where

$$\hat{q} = \begin{pmatrix} z & \epsilon \\ -\bar{\epsilon} & \bar{z} \end{pmatrix}. \quad (75)$$

The symbol \otimes denotes the Kronecker product. The blocks of the matrix \hat{X} can be identified with the Ginibre matrix and its Hermitian conjugate: $\hat{X}_{11} = X$, $\hat{X}_{22} = X^\dagger$ and $\hat{X}_{12} = \hat{X}_{21} = 0$, respectively. In order to calculate the propagator, we recall that the two-point correlations for the Ginibre matrix (14) with $\sigma^2 = 1/N$ are

$$\langle X_{ab} X_{cd}^\dagger \rangle = \int d\mu(X) X_{ab} X_{cd}^\dagger = \frac{1}{N} \delta_{ad} \delta_{bc} \quad (76)$$

and

$$\langle X_{ab} X_{cd} \rangle = \langle X_{ab}^\dagger X_{cd}^\dagger \rangle = 0. \quad (77)$$

Since all matrices have a block structure, it is convenient to separately write index positions of the blocks and positions of elements inside the blocks, and to split matrix indices into pairs of indices $A = (\alpha, a)$, $B = (\beta, b)$, $C = (\gamma, c)$, $D = (\delta, d)$, etc., with the Greek indices referring to the positions of the blocks, and small Latin indices to the positions within each block. The Greek indices run over the range 1 to 2 and the small Latin indices over the range 1 to N . The dimension of the matrices is $2N \times 2N$. This block structure is also inherited by the propagators. Using the identification $\hat{X}_{11} \leftrightarrow X$, $\hat{X}_{22} \leftrightarrow X^\dagger$, along with Eq. (76) and (77), we see that the propagator factorizes into the inter-block part (in Greek indices) and intra-block part (in Latin indices)

$$\hat{P}_{AB,CD} = \hat{p}_{\alpha\beta,\gamma\delta} \frac{1}{N} \delta_{ad} \delta_{bc}. \quad (78)$$

The only non-trivial elements of the inter-block part are $\hat{p}_{11,22} = \hat{p}_{22,11} = 1$. All remaining elements vanish: $\hat{p}_{\alpha\beta,\gamma\delta} = 0$. Since both the propagator (78) and the matrix $\hat{Q}_{AB} = q_{\alpha\beta} \delta_{ab}$ are proportional to Kronecker's deltas in Latin indices, this implies that the matrices \hat{G} and $\hat{\Sigma}$, being the solution of the Dyson-Schwinger equations (70), also are proportional to the Kronecker's delta in the intra-block indices

$$\hat{G}_{AB} = \hat{g}_{\alpha\beta} \delta_{ab}, \quad \hat{\Sigma}_{AB} = \hat{\sigma}_{\alpha\beta} \delta_{ab}. \quad (79)$$

Alternatively, one can write $\hat{G} = \hat{g} \otimes \mathbb{1}$ and $\hat{\Sigma} = \hat{\sigma} \otimes \mathbb{1}$. Therefore, one can reduce the Dyson-Schwinger equation (70) to equations for inter-block elements (in Greek indices)

$$\begin{aligned} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} &= \left(\begin{pmatrix} z & \epsilon \\ -\bar{\epsilon} & \bar{z} \end{pmatrix} - \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \right)^{-1}, \\ \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} &= \begin{pmatrix} 0 & g_{12} \\ g_{21} & 0 \end{pmatrix}. \end{aligned} \quad (80)$$

In the second equation, we used that $\hat{p}_{11,22} = \hat{p}_{22,11} = 1$ and $\hat{p}_{\alpha\beta,\gamma\delta} = 0$ for other combinations of indices. The limit $N \rightarrow \infty$ has already been taken in these equations, since they count contributions of planar diagrams. Now we can take the limit $\epsilon \rightarrow 0$ [cf. Eq. (65)]. This merely corresponds to setting $\epsilon = 0$. Eliminating the $\{\sigma_{\alpha\beta}\}$, we get

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} z & -g_{12} \\ -g_{21} & \bar{z} \end{pmatrix}^{-1}. \quad (81)$$

Setting $g = g_{11} = \bar{g}_{22}$ and $\gamma = g_{12} = -\bar{g}_{21}$ we obtain

$$\begin{pmatrix} g & \gamma \\ -\bar{\gamma} & \bar{g} \end{pmatrix} = \begin{pmatrix} z & \gamma \\ -\bar{\gamma} & \bar{z} \end{pmatrix}^{-1}. \quad (82)$$

The solution reads

$$g(z) = \frac{1}{z}, \quad \gamma(z) = 0 \quad \text{for } |z| \geq 1 \quad (83)$$

and

$$g(z) = \bar{z}, \quad |\gamma(z)| = \sqrt{1 - |z|^2} \quad \text{for } |z| \leq 1. \quad (84)$$

The solution for $\gamma(z)$ inside the unit circle is given up to the phase, but this is sufficient for our purposes since the correlations density $o(z)$ given by Eq. (68) depends only on the modulus of $\gamma(z)$. Using Eqs. (67) and (68), one eventually finds:

$$\rho(z) = \frac{1}{\pi} \chi_D(z) \quad (85)$$

and

$$o(z) = \frac{1}{\pi} (1 - |z|^2) \chi_D(z), \quad (86)$$

where χ_D is an indicator function for the unit disk, $\chi_D(z) = 1$ for $|z| \leq 1$ and $\chi_D(z) = 0$ for $|z| > 1$.

X. PRODUCT OF TWO GINIBRE MATRICES

In this section, we generalize the approach from the previous section to the product of two Ginibre matrices [28]. The integration measure for the product $X = X_1 X_2$ of independent Ginibre matrices X_1 and X_2 is the product of individual integration measures $d\mu(X_1)d\mu(X_2)$ given by Eq. (14). According to Eq. (76) the only non-vanishing two-point correlations are

$$\langle X_{1,ab} X_{1,cd}^\dagger \rangle = \langle X_{2,ab} X_{2,cd}^\dagger \rangle = \frac{1}{N} \delta_{ad} \delta_{bc} . \quad (87)$$

The Green's function (62) for the product reads

$$\widehat{G}(z, \epsilon) = \left\langle \left(\begin{pmatrix} z \mathbb{1}_N - X_1 X_2 & \epsilon \mathbb{1}_N \\ -\bar{\epsilon} \mathbb{1}_N & \bar{z} \mathbb{1}_N - X_2^\dagger X_1^\dagger \end{pmatrix} \right)^{-1} \right\rangle . \quad (88)$$

This form is difficult to handle because of the quadratic dependence on random matrices $X_1 X_2$. One can however linearize the problem by considering a block matrix of dimensions $2N \times 2N$

$$R = \begin{pmatrix} 0 & X_1 \\ X_2 & 0 \end{pmatrix} , \quad (89)$$

which we call root matrix because its square,

$$R^2 = \begin{pmatrix} X_1 X_2 & 0 \\ 0 & X_2 X_1 \end{pmatrix} , \quad (90)$$

reproduces two copies of the product, $X_1 X_2$ and $X_2 X_1$. The two copies have identical eigenvalues. The Green's function for the root matrix is

$$\widehat{G}(z, \epsilon) = \left\langle \left(\begin{pmatrix} z \mathbb{1}_N - R & \epsilon \mathbb{1}_N \\ -\bar{\epsilon} \mathbb{1}_N & \bar{z} \mathbb{1}_N - R^\dagger \end{pmatrix} \right)^{-1} \right\rangle , \quad (91)$$

which is actually a $4N \times 4N$ block matrix

$$\widehat{G}(z, \epsilon) = \left\langle \left(\widehat{q} \otimes \mathbb{1}_N - \widehat{R} \right)^{-1} \right\rangle , \quad (92)$$

where

$$\widehat{q} = \begin{pmatrix} z & 0 & \epsilon & 0 \\ 0 & z & 0 & \epsilon \\ -\bar{\epsilon} & 0 & \bar{z} & 0 \\ 0 & -\bar{\epsilon} & 0 & \bar{z} \end{pmatrix} \xrightarrow{\epsilon \rightarrow 0} \begin{pmatrix} z & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & \bar{z} & 0 \\ 0 & 0 & 0 & \bar{z} \end{pmatrix} \quad (93)$$

and

$$\widehat{R} = \begin{pmatrix} 0 & X_1 & 0 & 0 \\ X_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_2^\dagger \\ 0 & 0 & X_1^\dagger & 0 \end{pmatrix} . \quad (94)$$

In this representation, the resolvent (92) has the standard form in which \widehat{R} is linear in the random matrices X 's. Indexing blocks of \widehat{R} by $\widehat{R}_{\alpha\beta}$, with $\alpha = 1, \dots, 4$ and $\beta = 1, \dots, 4$, we have $\widehat{R}_{12} = X_1$, $\widehat{R}_{21} = X_2$, $\widehat{R}_{34} = X_2^\dagger$, $\widehat{R}_{43} = X_1^\dagger$. As follows from Eq. (87), the block \widehat{R}_{12} is correlated with \widehat{R}_{43} and \widehat{R}_{21} with \widehat{R}_{34} , so the propagator

$$\widehat{P}_{AB,CD} = \widehat{p}_{\alpha\beta,\gamma\delta} \frac{1}{N} \delta_{ad} \delta_{bc} \quad (95)$$

has the following non-zero elements, $\widehat{p}_{12,43} = \widehat{p}_{43,12} = \widehat{p}_{21,34} = \widehat{p}_{34,21} = 1$. All other elements of $\widehat{p}_{\alpha\beta,\gamma\delta} = 0$. The situation is completely analogous to that discussed in the previous section, except that now the problem has dimensions 4×4 in inter-block indices. The intra-block correlations are the same as before - that is they are proportional to $(1/N) \delta_{ad} \delta_{bc}$ - so the solution

has the diagonal form proportional to the Kronecker's delta in Latin indices (79). The Dyson-Schwinger equations (70) for the inter-block elements of the Green's function of the root matrix read for $\epsilon \rightarrow 0$

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} = \begin{pmatrix} z - \sigma_{11} & -\sigma_{12} & -\sigma_{13} & -\sigma_{14} \\ -\sigma_{21} & z - \sigma_{22} & -\sigma_{23} & -\sigma_{24} \\ -\sigma_{31} & -\sigma_{32} & \bar{z} - \sigma_{33} & -\sigma_{34} \\ -\sigma_{41} & -\sigma_{42} & -\sigma_{43} & \bar{z} - \sigma_{44} \end{pmatrix}^{-1} \quad (96)$$

and

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{pmatrix} = \begin{pmatrix} 0 & 0 & g_{24} & 0 \\ 0 & 0 & 0 & g_{13} \\ g_{42} & 0 & 0 & 0 \\ 0 & g_{31} & 0 & 0 \end{pmatrix}. \quad (97)$$

In the second equation, we used the propagator structure: $\hat{p}_{12,43} = \hat{p}_{43,12} = \hat{p}_{21,34} = \hat{p}_{34,21} = 1$ and $\hat{p}_{\alpha\beta,\gamma\delta} = 0$ otherwise. Inserting $\{\sigma_{\alpha\beta}\}$ into the first equation, we get

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} = \begin{pmatrix} z & 0 & -g_{24} & 0 \\ 0 & z & 0 & -g_{13} \\ -g_{42} & 0 & \bar{z} & 0 \\ 0 & -g_{31} & 0 & \bar{z} \end{pmatrix}^{-1}. \quad (98)$$

It is convenient to solve this equation by defining matrices \tilde{g} and $\tilde{\sigma}$ unitarily equivalent to \hat{g} and $\hat{\sigma}$: $\tilde{g} = P\hat{g}P^{-1}$ and $\tilde{\sigma} = P\hat{\sigma}P^{-1}$ where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (99)$$

The effect of the similarity transformation is equivalent to permutation of indices of the corresponding matrices: $g_{\alpha\beta} = \tilde{g}_{\pi(\alpha)\pi(\beta)}$ $\sigma_{\alpha\beta} = \tilde{\sigma}_{\pi(\alpha)\pi(\beta)}$ with $\pi : (1, 2, 3, 4) \rightarrow (1, 3, 2, 4)$. After this transformation, Eq. (98) is equivalent to

$$\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} & \tilde{g}_{13} & \tilde{g}_{14} \\ \tilde{g}_{21} & \tilde{g}_{22} & \tilde{g}_{23} & \tilde{g}_{24} \\ \tilde{g}_{31} & \tilde{g}_{32} & \tilde{g}_{33} & \tilde{g}_{34} \\ \tilde{g}_{41} & \tilde{g}_{42} & \tilde{g}_{43} & \tilde{g}_{44} \end{pmatrix} = \begin{pmatrix} z & -\tilde{g}_{34} & 0 & 0 \\ -\tilde{g}_{43} & \bar{z} & 0 & 0 \\ 0 & 0 & z & -\tilde{g}_{12} \\ 0 & 0 & -\tilde{g}_{21} & \bar{z} \end{pmatrix}^{-1}. \quad (100)$$

The matrix \tilde{g} is a block matrix made of 2×2 blocks. The off-diagonal blocks are zero while the diagonal ones fulfill the following equations

$$\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{21} & \tilde{g}_{22} \end{pmatrix} = \begin{pmatrix} z & -\tilde{g}_{34} \\ -\tilde{g}_{43} & \bar{z} \end{pmatrix}^{-1} \quad (101)$$

and

$$\begin{pmatrix} \tilde{g}_{33} & \tilde{g}_{34} \\ \tilde{g}_{43} & \tilde{g}_{44} \end{pmatrix} = \begin{pmatrix} z & -\tilde{g}_{12} \\ -\tilde{g}_{21} & \bar{z} \end{pmatrix}^{-1}. \quad (102)$$

The two equations have only a symmetric solution

$$\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{21} & \tilde{g}_{22} \end{pmatrix} = \begin{pmatrix} \tilde{g}_{33} & \tilde{g}_{34} \\ \tilde{g}_{43} & \tilde{g}_{44} \end{pmatrix} \quad (103)$$

being a solution of

$$\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} \\ \tilde{g}_{21} & \tilde{g}_{22} \end{pmatrix} = \begin{pmatrix} z & -\tilde{g}_{12} \\ -\tilde{g}_{21} & \bar{z} \end{pmatrix}^{-1}. \quad (104)$$

The last equation is exactly the same as for a single Ginibre matrix (81), so the solution eventually reads

$$\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} & \tilde{g}_{13} & \tilde{g}_{14} \\ \tilde{g}_{21} & \tilde{g}_{22} & \tilde{g}_{23} & \tilde{g}_{24} \\ \tilde{g}_{31} & \tilde{g}_{32} & \tilde{g}_{33} & \tilde{g}_{34} \\ \tilde{g}_{41} & \tilde{g}_{42} & \tilde{g}_{43} & \tilde{g}_{44} \end{pmatrix} = \begin{pmatrix} g & \gamma & 0 & 0 \\ -\bar{\gamma} & \bar{g} & 0 & 0 \\ 0 & 0 & g & \gamma \\ 0 & 0 & -\bar{\gamma} & \bar{g} \end{pmatrix} = \mathbb{1}_2 \otimes \begin{pmatrix} g & \gamma \\ -\bar{\gamma} & \bar{g} \end{pmatrix}, \quad (105)$$

where g and γ are given by Eqs. (83) and (84). If we permute indices back to the original order $\hat{g} = P^{-1}\tilde{g}P$, we find

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} = \begin{pmatrix} g & 0 & \gamma & 0 \\ 0 & g & 0 & \gamma \\ -\bar{\gamma} & 0 & \bar{g} & 0 \\ 0 & -\bar{\gamma} & 0 & \bar{g} \end{pmatrix} = \begin{pmatrix} g & \gamma \\ -\bar{\gamma} & \bar{g} \end{pmatrix} \otimes \mathbb{1}_2. \quad (106)$$

We see that the Green's function for the root matrix consists of two identical blocks equal to the Green's function of a single Ginibre matrix. In other words, the Green's function of the root matrix behaves exactly as a pair of copies of the Green's function of a single Ginibre matrix. The eigenvalue density and the growth rate of correlations between left and right eigenvectors of this matrix are given by Eqs. (85) and (86) as

$$\rho_R(z) = \frac{1}{\pi} \chi_D(z) \quad (107)$$

and

$$o_R(z) \sim \frac{1}{\pi} (1 - |z|^2) \chi_D(z). \quad (108)$$

Note that the size of the root matrix is $2N \times 2N$, so the leading term of the overlap behaves for large N as

$$O_R(z) \sim \frac{2N}{\pi} (1 - |z|^2) \chi_D(z). \quad (109)$$

From these expressions, one may derive the corresponding expressions for R^2 , which are directly related to the product $X_1 X_2$ as it stems from Eq. (90). The eigenvalues of R^2 are related to those of R as $\lambda = \lambda_R^2$, so one can find the densities by the change of variables $z = w^2$: $\rho(z) d^2 z = \rho_R(w) d^2 w$ and $O(z) d^2 z = O_R(w) d^2 w$. This gives

$$\rho(z) = \frac{1}{2\pi|z|} \chi_D(z) \quad (110)$$

and

$$O(z) \sim \frac{N}{\pi|z|} (1 - |z|) \chi_D(z), \quad (111)$$

respectively. The result for the eigenvalue density $\rho(z)$ was first found in [28]. The overlap $O(z)$ is a new result. The product $X_1 X_2$ is of size $N \times N$, so the growth rate is obtained by dividing $O(z)$ by N ,

$$o(z) = \lim_{N \rightarrow \infty} \frac{O(z)}{N} = \frac{1}{\pi|z|} (1 - |z|) \chi_D(z). \quad (112)$$

The radial profile is obtained from the last expression by setting $r = |z|$ and multiplying the result by $2\pi r$ [cf. Eq. (41)]. This gives a triangle law

$$o_{rad}(r) = 2(1 - r) \chi_I(r), \quad (113)$$

where χ_I is an indicator function for the interval $[0, 1]$: $\chi_I(r) = 1$ for $r \in [0, 1]$ and $\chi_I(r) = 0$ otherwise. This prediction is compared to Monte Carlo data for $N = 100$ in Fig. 5.

XI. PRODUCT OF ELLIPTIC GAUSSIAN MATRICES

For completeness, we also consider the product of elliptic matrices defined by the measure [89]

$$d\mu(X) = \frac{1}{Z} \exp \left[-\frac{1}{\sigma^2(1 - \kappa^2)} \text{Tr} \left(X X^\dagger - \frac{\kappa}{2} (X X + X^\dagger X^\dagger) \right) \right] DX. \quad (114)$$

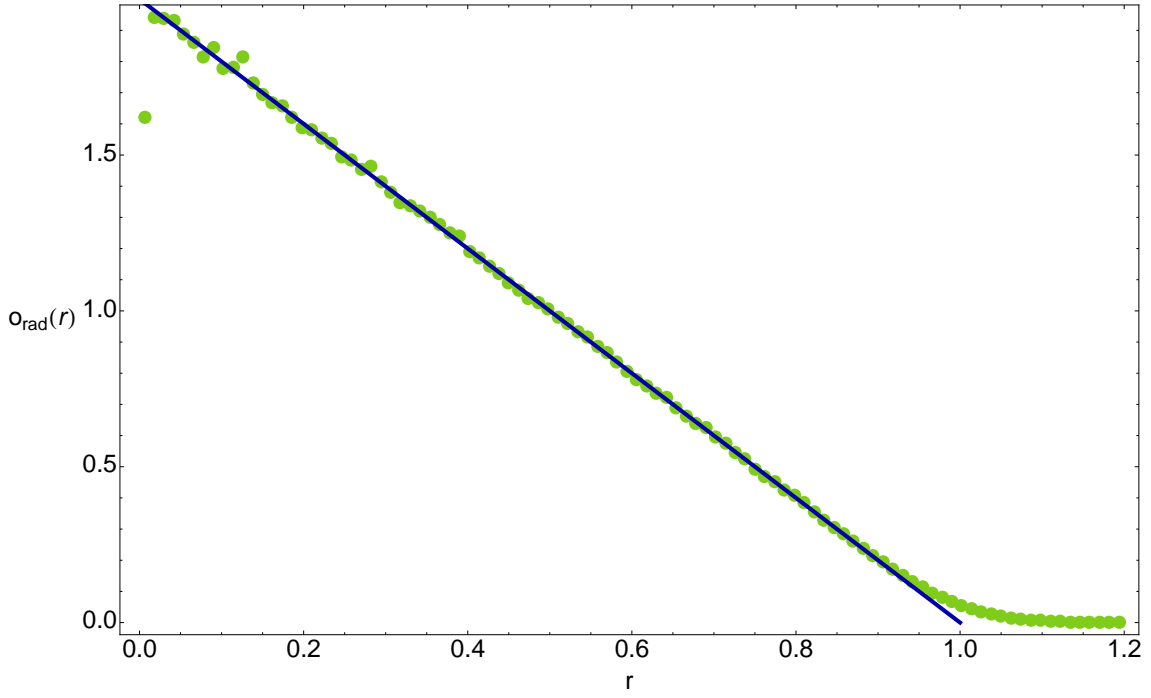


FIG. 5. Triangle law: theoretical prediction for $N \rightarrow \infty$ (113) and numerical histogram (points) generated in Monte Carlo simulations for 10^5 products of two 100×100 Ginibre matrices.

As before, we set $\sigma^2 = 1/N$ and scale it with N while taking the limit $N \rightarrow \infty$. The parameter κ belongs to the range $[-1, 1]$. It is related to the ellipse eccentricity. For $\kappa = 0$, (114) reproduces the Ginibre measure. Generically, the support of the eigenvalue density of matrices generated according to the measure given by Eq. (114) is elliptic. When κ approaches 1 (or -1), the support flattens and in the limit $\kappa \rightarrow 1$ gets completely squeezed to an interval of the real (or imaginary) axis. The corresponding matrix becomes Hermitian (or anti-Hermitian). The two-point correlations for the elliptic ensemble (114) are

$$\langle X_{ab} X_{cd}^\dagger \rangle = \langle X_{ab}^\dagger X_{cd} \rangle = \frac{1}{N} \delta_{ad} \delta_{bc} \quad (115)$$

and

$$\langle X_{ab} X_{cd} \rangle = \langle X_{ab}^\dagger X_{cd}^\dagger \rangle = \kappa \frac{1}{N} \delta_{ad} \delta_{bc} . \quad (116)$$

Consider the product $X = X_1 X_2$ of two elliptic matrices X_1 and X_2 with different eccentricity parameters κ_1 and κ_2 . As in the previous section, we construct the root matrix (94), which is a $4N \times 4N$ matrix. The propagator for the root matrix elements is

$$\hat{P}_{AB,CD} = \hat{p}_{\alpha\beta,\gamma\delta} \frac{1}{N} \delta_{ad} \delta_{bc} , \quad (117)$$

where $\hat{p}_{\alpha\beta,\gamma\delta}$ has now more nonzero elements. In addition to $\hat{p}_{12,43} = \hat{p}_{43,12} = \hat{p}_{21,34} = \hat{p}_{34,21} = 1$, we have $\hat{p}_{12,12} = \hat{p}_{21,21} = \kappa_1$ and $\hat{p}_{34,34} = \hat{p}_{43,43} = \kappa_2$, which come from Eq. (116). We can now write the Dyson-Schwinger equations for this propagator. The first equation is identical as that for the product of Ginibre matrices (96). The second one differs from the previous one (97), since now we have additional non-zero elements coming from the eccentricity parameters, which we denote κ_1 and κ_2 for the first and the second elliptic matrices in the product

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 g_{21} & g_{24} & 0 \\ \kappa_1 g_{12} & 0 & 0 & g_{13} \\ g_{42} & 0 & 0 & \kappa_2 g_{43} \\ 0 & g_{31} & \kappa_2 g_{34} & 0 \end{pmatrix} . \quad (118)$$

Inserting the $\{\sigma_{\alpha\beta}\}$ into Eq. (96) and permuting indices as in the previous section, we get

$$\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} & \tilde{g}_{13} & \tilde{g}_{14} \\ \tilde{g}_{21} & \tilde{g}_{22} & \tilde{g}_{23} & \tilde{g}_{24} \\ \tilde{g}_{31} & \tilde{g}_{32} & \tilde{g}_{33} & \tilde{g}_{34} \\ \tilde{g}_{41} & \tilde{g}_{42} & \tilde{g}_{43} & \tilde{g}_{44} \end{pmatrix} = \begin{pmatrix} z & -\tilde{g}_{34} & -\kappa_1 \tilde{g}_{31} & 0 \\ -\tilde{g}_{43} & \bar{z} & 0 & -\kappa_2 \tilde{g}_{42} \\ -\kappa_1 \tilde{g}_{13} & 0 & z & -\tilde{g}_{12} \\ 0 & -\kappa_2 \tilde{g}_{24} & -\tilde{g}_{21} & \bar{z} \end{pmatrix}^{-1} . \quad (119)$$

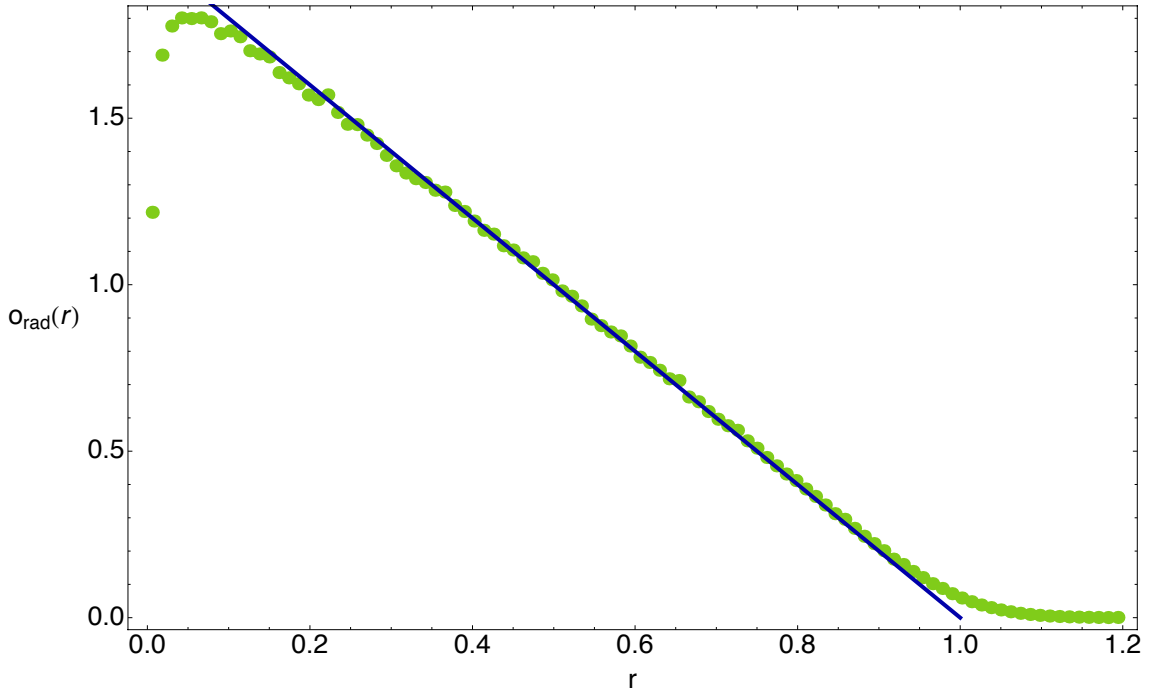


FIG. 6. Triangle law: theoretical prediction for $N \rightarrow \infty$ (113) and numerical histogram (points) generated in Monte Carlo simulations for 10^5 products of Ginibre times GUE matrices of dimensions 100×100 .

This equation is much more complicated than that for the product of Ginibre matrices (105), because the two off-diagonal blocks on the right-hand side are non-zero. However, making the *ansatz* that the off-diagonal blocks of the solution vanish

$$\begin{pmatrix} \tilde{g}_{13} & \tilde{g}_{14} \\ \tilde{g}_{23} & \tilde{g}_{24} \end{pmatrix} = \begin{pmatrix} \tilde{g}_{31} & \tilde{g}_{32} \\ \tilde{g}_{41} & \tilde{g}_{42} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (120)$$

forces the two remaining blocks to satisfy the very same equation as for the product of Ginibre matrices (100),

$$\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} & 0 & 0 \\ \tilde{g}_{21} & \tilde{g}_{22} & 0 & 0 \\ 0 & 0 & \tilde{g}_{33} & \tilde{g}_{34} \\ 0 & 0 & \tilde{g}_{43} & \tilde{g}_{44} \end{pmatrix} = \begin{pmatrix} z & -\tilde{g}_{34} & 0 & 0 \\ -\tilde{g}_{43} & \tilde{z} & 0 & 0 \\ 0 & 0 & z & -\tilde{g}_{12} \\ 0 & 0 & -\tilde{g}_{21} & \tilde{z} \end{pmatrix}^{-1}, \quad (121)$$

hence the solution is the same as before. This solution is independent of the eccentricity parameters κ_1 and κ_2 and moreover it is always spherically symmetric, even though the two matrices in the product are elliptic. To summarize, in the large N limit the eigenvalue density and the left-right eigenvector correlations for the product of two elliptic matrices are spherically symmetric (110) [28] and the eigenvector correlations are identical as for the product of Ginibre matrices (113). This prediction is compared to Monte Carlo data for $N = 100$ in Fig. 6. We see that it also follows the triangle law as for the product of Ginibre matrices. The finite N data exhibit however stronger finite size effects as compared to those for the product of two Ginibre matrices which manifest as a stronger deviation from the limiting density for small values of r . Compare Figs. 5 and 6.

XII. PRODUCT OF M GINIBRE MATRICES

We now proceed analogously as in Sec. X, where we discussed the product of two Ginibre matrices in the large N limit. The integration measure for the product $X = X_1 X_2 \cdots X_m$ of m independent Ginibre matrices X_1, X_2, \dots, X_m is the product $d\mu(X_1) d\mu(X_2) \cdots d\mu(X_m)$ of the individual integration measures given by Eq. (14). In turn, the two-point correlations are given by Eq. (76)

$$\langle X_{\mu,ab} X_{\nu,cd}^\dagger \rangle = \frac{1}{N} \delta_{\mu\nu} \delta_{ad} \delta_{bc}, \quad \langle X_{\mu,ab} X_{\nu,cd} \rangle = \langle X_{\mu,ab}^\dagger X_{\nu,cd}^\dagger \rangle = 0, \quad (122)$$

for $\mu, \nu = 1, \dots, m$ and $a, b, c, d = 1, \dots, N$. As in Sec. X, instead of directly applying the Green's function technique to the product $X_1 X_2 \cdots X_m$, we apply it to the root matrix R being a block matrix of dimensions $mN \times mN$

$$R = \begin{pmatrix} 0 & X_1 & 0 & \cdots & 0 \\ 0 & 0 & X_2 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & X_{m-1} \\ X_m & 0 & 0 & \cdots & 0 \end{pmatrix}. \quad (123)$$

The m -th power of the root matrix

$$R^m = \begin{pmatrix} X_1 X_2 \cdots X_m & 0 & \cdots & 0 \\ 0 & X_2 \cdots X_m X_1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & X_m X_1 \cdots X_{m-1} \end{pmatrix} \quad (124)$$

reproduces m cyclic copies of the product $X_1 X_2 \cdots X_m$, which all have identical eigenvalues. The Green's function for the root matrix is a $2mN \times 2mN$ block matrix

$$\hat{G}(z, \epsilon) = \left\langle \left(\hat{q} \otimes \mathbb{1}_N - \hat{R} \right)^{-1} \right\rangle, \quad (125)$$

where

$$\hat{q} = \begin{pmatrix} z \mathbb{1}_m & \epsilon \mathbb{1}_m \\ -\bar{\epsilon} \mathbb{1}_m & \bar{z} \mathbb{1}_m \end{pmatrix} \xrightarrow{\epsilon \rightarrow 0} \begin{pmatrix} z \mathbb{1}_m & 0 \\ 0 & \bar{z} \mathbb{1}_m \end{pmatrix} \quad (126)$$

and

$$\hat{R} = \begin{pmatrix} R & 0 \\ 0 & R^\dagger \end{pmatrix} = \begin{pmatrix} 0 & X_1 & 0 & \cdots & 0 \\ 0 & 0 & X_2 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & X_{m-1} \\ X_m & 0 & 0 & \cdots & 0 \\ & & & & 0 & 0 & \cdots & 0 & X_m^\dagger \\ & & & & X_1^\dagger & 0 & \cdots & 0 & 0 \\ & & & & 0 & X_2^\dagger & \cdots & 0 & 0 \\ & & & & & & \ddots & & \\ & & & & 0 & 0 & \cdots & X_{m-1}^\dagger & 0 \end{pmatrix}. \quad (127)$$

The resolvent given by Eq. (125) has the standard form with \hat{R} being linear in X 's. We index blocks of \hat{R} by Greek letters $\hat{R}_{\alpha\beta}$, with $\alpha, \beta = 1, \dots, 2m$. We have the following equivalence $\hat{R}_{\alpha, [\alpha]+1} \equiv X_\alpha$ and $\hat{R}_{m+[\alpha]+1, m+\alpha} \equiv X_\alpha^\dagger$ for $\alpha = 1, \dots, m$ and $[\alpha] = \alpha$ modulo m . All other blocks are zero. As follows from Eq. (122), we see that the only non-zero two-point correlations are

$$\langle R_{\alpha, [\alpha]+1} R_{m+[\alpha]+1, m+\alpha} \rangle = \langle X_\alpha X_\alpha^\dagger \rangle, \quad \langle R_{m+[\alpha]+1, m+\alpha} R_{\alpha, [\alpha]+1} \rangle = \langle X_\alpha^\dagger X_\alpha \rangle, \quad (128)$$

for $\alpha = 1, \dots, m$. Thus the propagator has the form

$$\hat{P}_{AB, CD} = \hat{p}_{\alpha\beta, \gamma\delta} \frac{1}{N} \delta_{ad} \delta_{bc}, \quad (129)$$

with

$$\hat{p}_{\alpha, [\alpha]+1; m+[\alpha]+1, m+\alpha} = \hat{p}_{m+[\alpha]+1, m+\alpha; \alpha, [\alpha]+1} = 1 \quad (130)$$

and $\hat{p}_{\alpha\beta, \gamma\delta} = 0$ otherwise. The situation is analogous to that discussed in Sec. X, except that now there are $2m \times 2m$ blocks. The intra-block correlations are the same as before $(1/N) \delta_{ad} \delta_{bc}$, so the solution is given as before as Kronecker product with

the Kronecker's delta in the intra-block indices $\hat{G}_{AB} = \hat{g}_{\alpha\beta} \delta_{ab}$ [cf. Eq. (79)]. The first Dyson-Schwinger equation (70) for the inter-block elements of the Green's function of the root matrix reads for $\epsilon \rightarrow 0$

$$\begin{pmatrix} g_{1,1} & \cdots & g_{1,2m} \\ \vdots & \ddots & \vdots \\ g_{2m,1} & \cdots & g_{2m,2m} \end{pmatrix} = \left[\begin{pmatrix} z \mathbb{1}_m & 0 \\ 0 & \bar{z} \mathbb{1}_m \end{pmatrix} - \begin{pmatrix} \sigma_{1,1} & \cdots & \sigma_{1,2m} \\ \vdots & \ddots & \vdots \\ \sigma_{2m,1} & \cdots & \sigma_{2m,2m} \end{pmatrix} \right]^{-1}. \quad (131)$$

The second Dyson-Schwinger equation (70) yields

$$\sigma_{\alpha, m+\alpha} = g_{[\alpha]+1, m+[\alpha]+1}, \quad \sigma_{m+[\alpha]+1, [\alpha]+1} = g_{m+\alpha, \alpha}, \quad (132)$$

for $\alpha = 1, \dots, m$, and $\sigma_{\alpha\beta} = 0$ for all other elements of the matrix $\hat{\sigma}$.

The Dyson-Schwinger equations assume a simple form in a modified basis obtained by permutation of matrix indices, $\alpha \rightarrow \pi(\alpha)$, where $\pi(\alpha) = 2\alpha - 1$ and $\pi(\alpha + m) = 2\alpha$ for $\alpha = 1, \dots, m$. We define $\hat{\sigma}_{\alpha\beta} = \tilde{\sigma}_{\pi(\alpha)\pi(\beta)}$ and $\hat{g}_{\alpha\beta} = \tilde{g}_{\pi(\alpha)\pi(\beta)}$. This transformation can be alternatively viewed as a similarity transformation $\hat{g} = P^{-1} \tilde{g} P$ and $\hat{\sigma} = P^{-1} \tilde{\sigma} P$, where the elements of the matrix P are $P_{\alpha\beta} = \delta_{\alpha\pi(\beta)}$ and $P_{\alpha\beta}^{-1} = \delta_{\pi(\alpha)\beta}$. Clearly, \tilde{g} and \hat{g} as well as $\tilde{\sigma}$ and $\hat{\sigma}$ are unitarily equivalent. Equations (132) are equivalent to

$$\tilde{\sigma}_{2\alpha-1, 2\alpha} = \tilde{g}_{(2\alpha+1), (2\alpha+2)}, \quad \sigma_{2\alpha, 2\alpha-1} = \tilde{g}_{(2\alpha-2), (2\alpha-3)}, \quad (133)$$

where the function $y = (x)$ on the right hand side maps the set of integers on the subset $\{1, 2, \dots, 2m\}$ in the following way. Any integer x can be decomposed uniquely as $x = y + 2mk$ where $y \in \{1, 2, \dots, 2m\}$ and k is an integer. The function (x) selects y from this decomposition. In particular $(x) = x$ and $(2m+1) = 1$, $(2m+2) = 2$, $(0) = 2m$, $(-1) = 2m-1$. Eliminating $\tilde{\sigma}$'s from the Dyson-Schwinger equations, we obtain a compact equation for \tilde{g} 's

$$\begin{pmatrix} \tilde{g}_{11} & \tilde{g}_{12} & \tilde{g}_{13} & \tilde{g}_{14} & \cdots & \cdots & \cdots \\ \tilde{g}_{21} & \tilde{g}_{22} & \tilde{g}_{23} & \tilde{g}_{24} & \cdots & \cdots & \cdots \\ \tilde{g}_{31} & \tilde{g}_{32} & \tilde{g}_{33} & \tilde{g}_{34} & \cdots & \cdots & \cdots \\ \tilde{g}_{41} & \tilde{g}_{42} & \tilde{g}_{43} & \tilde{g}_{44} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \tilde{g}_{2m-1, 2m-1} & \tilde{g}_{2m, 2m-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \tilde{g}_{2m-1, 2m} & \tilde{g}_{2m, 2m} \end{pmatrix} = \begin{pmatrix} z & -\tilde{g}_{34} & 0 & 0 & \cdots & 0 & 0 \\ -\tilde{g}_{2m, 2m-1} & \bar{z} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & z & -\tilde{g}_{45} & \cdots & 0 & 0 \\ 0 & 0 & -\tilde{g}_{12} & \bar{z} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & z & -\tilde{g}_{12} \\ 0 & 0 & 0 & 0 & 0 & -\tilde{g}_{2m-2, 2m-3} & \bar{z} \end{pmatrix}^{-1}. \quad (134)$$

The matrix \tilde{g} can be viewed as a block matrix made of 2×2 blocks. The off-diagonal blocks are zero and the diagonal ones fulfill the following equations

$$\begin{pmatrix} \tilde{g}_{2\alpha-1, 2\alpha-1} & \tilde{g}_{2\alpha-1, 2\alpha} \\ \tilde{g}_{2\alpha, 2\alpha-1} & \tilde{g}_{2\alpha, 2\alpha} \end{pmatrix} = \begin{pmatrix} z & -\tilde{g}_{(2\alpha+1), (2\alpha+2)} \\ -\tilde{g}_{(2\alpha-2), (2\alpha-3)} & \bar{z} \end{pmatrix}^{-1}, \quad (135)$$

for $\alpha = 1, \dots, m$. Making the *ansatz* that the solution should be symmetric - that is $\tilde{g}_{2\alpha-1, 2\alpha-1} = g$, $\tilde{g}_{2\alpha, 2\alpha} = \bar{g}$, $\tilde{g}_{2\alpha-1, 2\alpha} = \gamma$ and $\tilde{g}_{2\alpha, 2\alpha-1} = -\bar{\gamma}$ for all $\alpha = 1, \dots, m$, the last equations reduce to a single one

$$\begin{pmatrix} g & \gamma \\ -\bar{\gamma} & \bar{g} \end{pmatrix} = \begin{pmatrix} z & \gamma \\ -\bar{\gamma} & \bar{z} \end{pmatrix}^{-1}, \quad (136)$$

which is identical as that for a single Ginibre matrix (82). Hence, the solution for γ and g is given by Eqs. (83) and (84). This *ansatz* is equivalent to the one we used for $m = 2$ and it merely means that the solution should not break the symmetry between different cyclic permutations of Ginibre matrices in the product. Inserting the solution into \tilde{g} we find

$$\tilde{g} = \mathbb{1}_m \otimes \begin{pmatrix} g & \gamma \\ -\bar{\gamma} & \bar{g} \end{pmatrix}, \quad (137)$$

where g and γ are given by Eqs. (83) and (84). Permuting indices back to the original order $\hat{g} = P \tilde{g} P^{-1}$

$$\hat{g} = \begin{pmatrix} g & \gamma \\ -\bar{\gamma} & \bar{g} \end{pmatrix} \otimes \mathbb{1}_m. \quad (138)$$

Hence, we see that the Green's function of the root matrix behaves as m copies of the Green's function of a single Ginibre matrix. The eigenvalue density and the growth rate of correlations between left and right eigenvectors of this matrix are identical as Eqs. (85) and (86), namely

$$\rho_R(z) = \frac{1}{\pi} \chi_D(z) \quad (139)$$

and

$$o_R(z) = \frac{1}{\pi}(1 - |z|^2)\chi_D(z). \quad (140)$$

The leading term of the overlap is therefore

$$O_R(z) \sim \frac{mN}{\pi}(1 - |z|^2)\chi_D(z). \quad (141)$$

The eigenvalues λ of R^m are related to those of R as $\lambda = \lambda_R^m$, so by changing variables as $z = w^m$ we can find the corresponding distributions for R^m : $\rho(z)d^2z = \rho_R(w)d^2w$ and $O(z)d^2z = O_R(w)d^2w$. This gives

$$\rho(z) = \frac{1}{m\pi}|z|^{\frac{2}{m}-2}\chi_D(z) \quad (142)$$

and

$$O(z) \sim \frac{N}{\pi}|z|^{\frac{2}{m}-2}\left(1 - |z|^{\frac{2}{m}}\right)\chi_D(z), \quad (143)$$

respectively. Thus for large N the growth rate of the overlap for the product $X_1 X_2 \cdots X_m$ is

$$o(z) = \lim_{N \rightarrow \infty} \frac{O(z)}{N} = \frac{1}{\pi}|z|^{\frac{2}{m}-2}\left(1 - |z|^{\frac{2}{m}}\right)\chi_D(z). \quad (144)$$

The radial profile defined by Eq. (41) is

$$o_{rad}(r) = 2r^{\frac{2}{m}-1}\left(1 - r^{\frac{2}{m}}\right)\chi_I(r), \quad (145)$$

where as before χ_D is the indicator function for the unit disk $|z| \leq 1$ and χ_I for the interval $[0, 1]$. While finalizing the manuscript, we learned that this result was derived independently in [90] with the aid of an extension of the Haagerup-Larsen theorem [91, 92]. The integrated growth rate is

$$\int o_{rad}(r)dr = \frac{m}{2}, \quad (146)$$

which means that for large N the overlap grows as $O \sim mN/2$ in agreement with Eq. (59). In Fig. 7, we plot the expression given by Eq. (145) for $m = 4$ and compare it to Monte Carlo data for $N = 100$.

XIII. CONCLUSIONS

In this paper, we have studied macroscopic and microscopic eigenvector statistics of the product of Ginibre matrices. We have developed analytic methods to calculate the left-right eigenvector overlap for finite N and in the limit $N \rightarrow \infty$. The overlap is not only an interesting object from the mathematical point of view but is also of interest for physical problems. In the physics literature, it is known as Petermann factor and is for example used as a measure of non-orthogonality of cavity modes in chaotic scattering [93, 94]. It plays also an important role in the description of Dysonian diffusion for non-Hermitian random matrices [95, 96].

There are many open problems and potential generalizations of the studies presented in this paper. For example, one may try to extend the studies of the microscopic eigenvector statistics to products of truncated unitary matrices [97], which can also be mapped onto a determinantal point process [51] via generalized Schur decomposition [45]. A great challenge is to determine the microscopic eigenvalue and eigenvector statistics for products of elliptic matrices or to find any non-trivial solvable example of products of random matrices having non-spherical measures.

Concerning the large N limit and macroscopic statistics, it would be interesting to generalize the calculations of the overlap to polynomials of random matrices [37, 40, 42] and to go beyond isotropic (R-diagonal) matrices [90, 92], as well as to better understand the overlap in terms of the quaternionic formalism [44], and finally to calculate the off-diagonal elements of the overlap (7) using the Bethe-Salpeter equation [79].

XIV. ACKNOWLEDGMENTS

We would like to thank Romuald Janik for many interesting discussions. P.V. acknowledges the stimulating research environment provided by the EPSRC Centre for Doctoral Training in Cross-Disciplinary Approaches to Non-Equilibrium Systems (CANES, Grant No. EP/L015854/1).

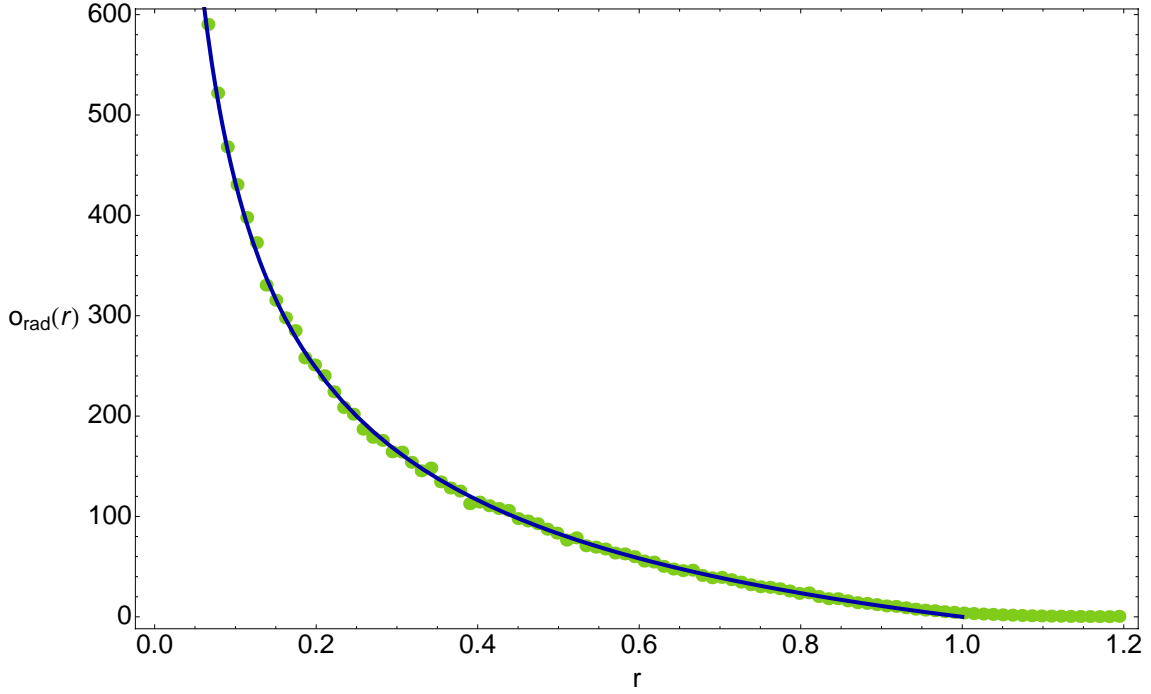


FIG. 7. Limiting overlap density for $m = 4$: theoretical prediction for $N \rightarrow \infty$ (145) and numerical histogram (points) generated in Monte Carlo simulations for 10^5 products of four 100×100 Ginibre matrices.

Appendix A: Calculation of the integral (57)

In this Appendix, we detail the calculation of the integral given by Eq. (57)

$$O = \frac{1}{Z} \int \prod_{\alpha=1}^{N-1} \left(1 + \frac{1}{|\lambda_N - \lambda_\alpha|^2} \right) |\Delta_N(\boldsymbol{\lambda})|^2 \prod_{\alpha=1}^N e^{-|\lambda_\alpha|^2} d^2 \lambda_\alpha, \quad (\text{A1})$$

where we have renamed the Vandermonde determinant on N complex variables as $\Delta_N(\boldsymbol{\lambda})$ for convenience.

We can rewrite this as

$$O = \frac{1}{Z} \int \prod_{\alpha=1}^{N-1} \left(\frac{|\lambda_N - \lambda_\alpha|^2 + 1}{|\lambda_N - \lambda_\alpha|^2} \right) |\Delta_N(\boldsymbol{\lambda})|^2 \prod_{\alpha=1}^N e^{-|\lambda_\alpha|^2} d^2 \lambda_\alpha = \frac{1}{Z} \int \prod_{\alpha=1}^{N-1} (|\lambda_N - \lambda_\alpha|^2 + 1) |\Delta_{N-1}(\boldsymbol{\lambda})|^2 \prod_{\alpha=1}^N e^{-|\lambda_\alpha|^2} d^2 \lambda_\alpha, \quad (\text{A2})$$

which can be more compactly expressed as

$$O = \frac{1}{Z} (N-1)! \int d^2 \lambda_N e^{-|\lambda_N|^2} \det \left(\underbrace{\int d^2 z e^{-|z|^2} z^{j-1} \bar{z}^{k-1} (|\lambda_N - z|^2 + 1)}_{I_{jk}(\lambda_N)} \right)_{j,k=1,\dots,N-1}, \quad (\text{A3})$$

using the complex version of Andréief's identity [98]. The integral over z yields

$$I_{jk}(\lambda) = \pi (|\lambda|^2 + 1)(k-1)! + k! \delta_{j,k} - \pi k! \lambda \delta_{j-1,k} - \pi (k-1)! \bar{\lambda} \delta_{j+1,k}. \quad (\text{A4})$$

This is a tridiagonal matrix. When calculating its determinant $I_{N-1}(\lambda) = \det (I_{jk}(\lambda))_{j,k=1,\dots,N-1}$ it is convenient to pull out a common factor from each column of the matrix

$$I_{jk}(\lambda) = \pi (k-1)! D_{jk}(\lambda), \quad (\text{A5})$$

where

$$D_{jk}(\lambda) = (|\lambda|^2 + 1 + k) \delta_{j,k} - k \lambda \delta_{j-1,k} - \bar{\lambda} \delta_{j+1,k}. \quad (\text{A6})$$

The determinant $I_{N-1}(\lambda)$ can be related to the determinant $D_{N-1}(\lambda) = \det(D_{jk}(\lambda))_{j,k=1,\dots,N-1}$ as follows

$$I_{N-1}(\lambda) = \pi^{N-1} 0!1! \cdots (N-2)! D_{N-1}(\lambda) . \quad (\text{A7})$$

Thus we can rewrite (A3) as

$$O = \frac{1}{\pi N!} \int d^2\lambda e^{-|\lambda|^2} D_{N-1}(\lambda) , \quad (\text{A8})$$

where we have also replaced the normalization constant by the explicit expression $Z = \pi^N 1!2! \cdots N!$ [cf. Eq. (24)]. It remains to find the determinant $D_n(\lambda)$ for $n = N - 1$. It has the form

$$D_n = \begin{vmatrix} a_1 & b_1 & & 0 \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ 0 & & c_{n-1} & a_n \end{vmatrix} , \quad (\text{A9})$$

with $a_n = |\lambda|^2 + 1 + n$, $b_n = -n\lambda$, $c_n = -\bar{\lambda}$. In general the sequence $\{D_n\}$ is called *continuant* and satisfies the following recurrence relation

$$D_n = a_n D_{n-1} - b_{n-1} c_{n-1} D_{n-2} ,$$

with initial conditions $D_0 = 1$ and $D_1 = a_1$. In our case the recurrence takes the form

$$D_n = (|\lambda|^2 + 1 + n) D_{n-1} - n |\lambda|^2 D_{n-2} . \quad (\text{A10})$$

The sequence $\{D_n\}$ reveals an interesting pattern for small n which allows us to conjecture that D_n is given in closed form by

$$D_n(\lambda) = \sum_{k=0}^n \frac{n!(n+1-k)}{k!} |\lambda|^{2k} . \quad (\text{A11})$$

One can check by straightforward algebraic manipulations that this polynomial indeed fulfills the recurrence relation (A10). The Gaussian integral of this polynomial gives a simple result

$$\int d^2\lambda e^{-|\lambda|^2} D_n(\lambda) = \pi \sum_{k=0}^n n!(n+1-k) = \pi n! \frac{(n+1)(n+2)}{2} = \pi(n+1)! \left(1 + \frac{n}{2}\right) , \quad (\text{A12})$$

which for $n = N - 1$, using (A8), leads to

$$O = 1 + \frac{1}{2}(N-1) , \quad (\text{A13})$$

as claimed in (58).

-
- [1] H. Furstenberg and H. Kesten, *Ann. Math. Statist.* **31**, 457 (1960).
 - [2] V. I. Oseledec, *Trans. Moscow Math. Soc.* **19**, 197 (1968).
 - [3] P. Bougerol and J. Lacroix, *Products of Random Matrices with Applications to Schrödinger Operators*, (Birkhäuser, Basel, 1985).
 - [4] J. E. Cohen, H. Kesten and C. M. Newman (eds), *Random matrices and their applications*, *Contemporary Mathematics* **50**, (Providence, RI: American Mathematical Society, 1986).
 - [5] A. Crisanti, G. Paladin and A. Vulpiani, *Products of random matrices, Random matrices and their applications*, (Springer-Verlag, Berlin Heidelberg, 1993).
 - [6] D. Ruelle, *Invent. Math.* **34**, 231 (1976).
 - [7] B. Derrida and H. J. Hilhorst, *J. Phys. A* **16**, 2641 (1983).
 - [8] C. de Calan, J. M. Luck, T. M. Nieuwenhuizen and D. Petritis, *J. Phys. A* **18**, 501 (1985).
 - [9] J. M. Luck, *Systèmes désordonnés unidimensionnels*, *Collection Aléa-Saclay* (Commissariat à l'énergie atomique, Gif-sur-Yvette, 1992).
 - [10] E. Gudowska-Nowak, R. A. Janik, J. Jurkiewicz, M. A. Nowak and W. Wiecezorek, *Chem. Phys.* **375**, 380 (2010).
 - [11] T. Guhr, A. Müller-Groeling and H. A. Weidenmüller, *Phys. Rep.* **299**, 189 (1998).

- [12] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
- [13] C. W. J. Beenakker, Rev. Mod. Phys. **69**, 731 (1997).
- [14] Y. Ephraim and N. Merhav, IEEE T. Inform. Theory **48**, 1518 (2002).
- [15] A. D. Jackson, B. Lautrup, P. Johansen and M. Nielsen, Phys. Rev. E **66**, 066124 (2002).
- [16] R. Lohmayer, H. Neuberger and T. Wettig, JHEP **0905**, 107 (2009).
- [17] R. R. Mueller, IEEE Trans. Inf. Theor. **48**, 2086 (2002).
- [18] R. Couillet and M. Debbah, *Random Matrix Methods for Wireless Communications*, (Cambridge University Press, 2011).
- [19] M. Potters, J.-P. Bouchaud and L. Laloux, Acta Phys. Pol. B **36**, 2767 (2005).
- [20] J.-P. Bouchaud, L. Laloux, M. A. Miceli and M. Potters, Eur. Phys. J. B **2**, 201 (2007).
- [21] Z. Burda, A. Jarosz, J. Jurkiewicz, M. A. Nowak, G. Papp and I. Zahed, Quant. Financ. **11**, 1103 (2011).
- [22] G. Akemann, J. Baik and P. Di Francesco (Ed.), *The Oxford Handbook of Random Matrix Theory*, (Oxford University Press, Oxford, 2011).
- [23] D. V. Voiculescu, J. Operator Theory **18**, 223 (1987).
- [24] D. V. Voiculescu, K. J. Dykema and A. Nica, *Free random variables*, CRM Monograph Series 1, (Providence, RI: American Mathematical Society, 1992).
- [25] R. A. Janik and W. Wiecek, J. Phys. A **37**, 6521 (2004).
- [26] V. Kargin, Probability Theory and Related Fields, **139**, 397 (2007).
- [27] Z. Burda, A. Jarosz, G. Livan, M. A. Nowak and A. Swiech, Phys. Rev. E **82**, 061114 (2010).
- [28] Z. Burda, R. A. Janik and B. Wacław, Phys. Rev. E **81**, 041132 (2010).
- [29] N. Alexeev, F. Götze and A. Tikhomirov, *On the asymptotic distribution of singular values of products of large rectangular random matrices*, Preprint [arXiv:1012.2586] (2010).
- [30] F. Götze and A. Tikhomirov, *On the Asymptotic Spectrum of Products of Independent Random Matrices*, Preprint [arXiv:1012.2710] (2010).
- [31] Z. Burda, R. A. Janik and M. A. Nowak, Phys. Rev. E **84**, 061125 (2011).
- [32] S. O'Rourke and A. Soshnikov, Electron. J. Probab. **16**, 2219 (2011).
- [33] K. A. Penson and K. Życzkowski, Phys. Rev. E **83**, 061118 (2011).
- [34] Z. Burda, M. A. Nowak and A. Swiech, Phys. Rev. E **86**, 061137 (2012).
- [35] Z. Burda, G. Livan and A. Swiech, Phys. Rev. E **88**, 022107 (2013).
- [36] Z. Burda, Conf. Ser. **473**, 012002 (2013).
- [37] S. Belinschi, T. Mai and R. Speicher, *Analytic subordination theory of operator-valued free additive convolution and the solution of a general random matrix problem*, Preprint [arXiv:1303.3196] (2013).
- [38] F. Götze, A. Naumov and A. Tikhomirov, *Distribution of Linear Statistics of Singular Values of the Product of Random Matrices*, Preprint [arXiv:1412.3314] (2014).
- [39] S. O'Rourke, D. Renfrew, A. Soshnikov and V. Vu, J. Stat. Phys. **160**, 89 (2015).
- [40] S. T. Belinschi, R. Speicher, J. Treilhard and C. Vargas, Int. Math. Res. Not. **14**, 5933 (2015).
- [41] F. Götze, H. Kösters and A. Tikhomirov, Random Matrices: Theory Appl. **4**, 1550005 (2015).
- [42] S. T. Belinschi, P. Sniady and R. Speicher, *Eigenvalues of non-hermitian random matrices and Brown measure of non-normal operators: hermitian reduction and linearization method*, Preprint [arXiv:1506.02017] (2015).
- [43] R. Speicher, Acta Phys. Pol. B **46**, 1611 (2015).
- [44] Z. Burda and A. Swiech, Phys. Rev. E **92**, 052111 (2015).
- [45] G. Akemann and Z. Burda, J. Phys. A **45**, 465201 (2012).
- [46] G. Akemann, M. Kieburg and L. Wei, J. Phys. A **46**, 275205 (2013).
- [47] G. Akemann and E. Strahov, J. Stat. Phys. **151**, 987 (2013).
- [48] G. Akemann, J. R. Ipsen and M. Kieburg, Phys. Rev. E **88**, 052118 (2013).
- [49] J. R. Ipsen, J. Phys. A **46**, 265201 (2013).
- [50] A. Lakshminarayan, J. Phys. A **46**, 152003 (2013).
- [51] G. Akemann, Z. Burda, M. Kieburg and T. Nagao, J. Phys. A **47**, 255202 (2014).
- [52] G. Akemann, J. R. Ipsen and E. Strahov, Random Matrices: Theory Appl. **03**, 1450014 (2014).
- [53] P. J. Forrester, J. Phys. A **47**, 065202 (2014).
- [54] P. J. Forrester, J. Phys. A **47**, 345202 (2014).
- [55] J. R. Ipsen and M. Kieburg, Phys. Rev. E **89**, 032106 (2014).
- [56] A. B. J. Kuijlaars and D. Stivigny, Random Matrices: Theory Appl. **3**, 1450011 (2014); A. B. J. Kuijlaars and L. Zhang, Comm. Math. Phys. **332**, 759 (2014).
- [57] D.-Z. Liu and Y. Wang, *Universality for products of random matrices I: Ginibre and truncated unitary cases*, Preprint [arXiv:1411.2787] (2014).
- [58] T. Neuschel, Random Matrices: Theory Appl. **3**, 1450003 (2014).
- [59] T. Claeys, A. B. J. Kuijlaars and D. Wang, *Correlation kernels for sums and products of random matrices*, Preprint [arXiv:1505.00610] (2015).
- [60] S. Hameed, K. Jain and A. Lakshminarayan, J. Phys. A **48**, 385204 (2015).
- [61] S. Kumar, *Exact evaluations of some Meijer G-functions and probability of all eigenvalues real for product of two Gaussian matrices*, Preprint [arXiv:1507.05571] (2015).
- [62] G. Akemann and J. R. Ipsen, Acta Phys. Pol. B **46**, 1747 (2015).
- [63] M. Kieburg, A. B. J. Kuijlaars and D. Stivigny, *Singular value statistics of matrix products with truncated unitary matrices*, Preprint [arXiv:1501.03910] (2015).

- [64] J. R. Ipsen, *Products of Independent Gaussian Random Matrices*, Preprint [arXiv:1510.06128] (2015).
- [65] K. Adhikari, N. K. Reddy, T. R. Reddy and K. Saha, Ann. Inst. H. Poincaré Probab. Statist. **52**, 16 (2016).
- [66] M. Kieburg and H. Kösters, *Exact Relation between Singular Value and Eigenvalue Statistics*, Preprint [arXiv:1601.02586] (2016).
- [67] C. M. Newman, Commun. Math. Phys. **103**, 121 (1986).
- [68] M. Isopi and C. M. Newman, Comm. Math. Phys. **143**, 591 (1992).
- [69] Z.-Q. Bai, J. Phys. A **40**, 8315 (2007).
- [70] V. Kargin, J. Funct. Anal. **255**, 1874 (2008).
- [71] M. Pollicott, Invent. Math. **181**, 209 (2010).
- [72] J. Vanneste, Phys. Rev. E **81**, 036701 (2010).
- [73] P. J. Forrester, J. Stat. Phys. **151**, 796 (2013).
- [74] V. Kargin, J. Stat. Phys. **157**, 70 (2014).
- [75] G. Akemann, Z. Burda and M. Kieburg, J. Phys. A **47**, 395202 (2014).
- [76] J. R. Ipsen, J. Phys. A **48**, 155204 (2015).
- [77] P. J. Forrester, J. Phys. A, **48**, 215205 (2015).
- [78] J. T. Chalker and B. Mehlig, Phys. Rev. Lett. **81**, 3367 (1998).
- [79] B. Mehlig and J. T. Chalker, J. Math. Phys. **41**, 3233 (2000).
- [80] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, (Academic Press, New York, 2000).
- [81] R. A. Janik, Ph.D. Thesis, Jagiellonian University (Kraków 1996), unpublished.
- [82] R. A. Janik, M. A. Nowak, G. Papp and I. Zahed, Nucl. Phys. B **501**, 603 (1997).
- [83] R. A. Janik, M. A. Nowak, G. Papp, J. Wambach and I. Zahed, Phys. Rev. E **55**, 4100 (1997).
- [84] R. A. Janik, W. Nörenberg, M. A. Nowak, G. Papp and I. Zahed, Phys. Rev. E **60**, 2699 (1999).
- [85] J. Ginibre, J. Math. Phys. **6**, 440 (1965).
- [86] G. 't Hooft, Nucl. Phys. B **72**, 461 (1974).
- [87] E. Brezin, C. Itzykson, G. Parisi and J.-B. Zuber, Commun. Math. Phys. **59**, 35 (1978).
- [88] D. Bessis, C. Itzykson and J.-B. Zuber, Adv. Appl. Math. **1**, 109 (1980).
- [89] V. L. Girko, Theor. Probab. Appl. (USSR) **30**, 640 (1985).
- [90] S. T. Belinschi, M. A. Nowak, R. Speicher and W. Tarnowski, *Mean eigenvalue condition numbers and eigenvector correlations from the single ring theorem*, Preprint [arXiv:1608.04923] (2016).
- [91] J. Feinberg and A. Zee, Nucl. Phys. B **504**, 579 (1997).
- [92] U. Haagerup and F. Larsen, J. Funct. Anal. **176**, 331 (2000).
- [93] K. Frahm, H. Schomerus, M. Patra and C. W. J. Beenakker, Europhys. Lett. **49**, 48 (2000).
- [94] M. V. Berry, J. Modern Optics **50**, 63 (2003).
- [95] Z. Burda, J. Grela, M. A. Nowak, W. Tarnowski and P. Warchol, Phys. Rev. Lett. **113**, 104102 (2014).
- [96] Z. Burda, J. Grela, M. A. Nowak, W. Tarnowski and P. Warchol, Nucl. Phys. B **897**, 421 (2015).
- [97] K. Życzkowski and H.-J. Sommers, J. Phys. A **33**, 2045 (2000).
- [98] C. Andréief, Mém. de la Soc. Sci. de Bordeaux **2**, 1 (1883).